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A Surrogate-Based Multiobjective Metaheuristic and Network Degradation Simulation Model for Robust Toll Pricing

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ABSTRACT

From a problem-centric perspective, robust network design problems are systems engineering methods that share common research gaps. First, problem sizes are constrained due to the use of multi-objective solution algorithms that are notoriously inefficient due to costly function evaluations. Second, link disruptions at a network level are not easy to model realistically. In this paper a stochastic search metaheuristic based on radial basis functions is proposed for constrained multiobjective problems. It is proven to converge, and compared with conventional metaheuristics for four test problems. A scenario simulation method based on multivariate Bernoulli random variables that accounts for correlations between link failures is proposed to sample scenarios for a mean-variance toll pricing problem. Four tests are conducted on the classical Sioux Falls network to gain some insights into the algorithm, the simulation model, and to the robust toll pricing problem. The first test empirically measures the efficiency of the simulation algorithm and approximate Pareto set by obtaining a standard error in the ϵ-indicator measure for a given number of scenarios and iterations. The second test compares the dominance of the proposed heuristic's solutions with a conventional multiobjective genetic algorithm by comparing the average ϵ-indicator. The third test quantifies the gap due to falsely assuming that link failures are independent of each other when they are not. The last test quantifies the value of having the flexibility to adapt a Pareto set of toll pricing solutions to changing probability regimes such as peak and off-peak hurricane or snow seasons.

Keywords: multiobjective, surrogate model, radial basis function, robust optimization, multivariate Bernoulli, toll pricing problem, network design

1. BACKGROUND

Network design in transportation covers a wide range of strategies and operations, including road expansion, signal control, ramp metering, and toll pricing (Yang and Bell, 1997). With each strategy, a number of different objectives can be considered such as improving system travel times versus improving social equity. As public policymakers face a growing need to consider more complex solutions such as sustainability and performance under disruptions, the need for effectively solving multi-objective network design problems will only increase with time.

Among the multiobjective transportation network design problems, those considered under uncertainty have garnered much interest from researchers. When operating or planning a network that is subject to randomness or stochastic network disruptions, decision-makers may prefer solutions that are resilient against the uncertainty. Solutions to optimization problems that are least sensitive to uncertainty are called robust solutions.

Robust network design problems have been studied quite extensively in the transportation literature. A robust optimization problem can be formulated a number of different ways, one of which is a mean-variance formulation that maximizes the expected value and minimizes the variance of the objective value as defined in Mulvey *et al*. (1995). Chen *et al*. (2006) use a genetic algorithm (GA) to solve a robust toll pricing problem using this mean-variance formulation that maximizes expected profit or social welfare while minimizing the variance. Sharma *et al*. (2009) solve a robust discrete network design problem formulated as a mean-variance problem using a GA. Karoonsoontawong and Waller (2007) apply the robust mean-variance approach to the continuous network design problem with dynamic traffic assignment using cell transmission. Yin (2008) proposes a robust approach to optimal traffic signal timing using three common robust optimization methods: the mean-variance method, minimizing the conditional value at risk, and the min-max approach. Further, Yin *et al*. (2009) apply scenario based, sensitivity based and min-max optimization based approaches to general road network improvement problems under demand uncertainty. These studies suggest a number of different robust optimization approaches have been implemented in the past, with a preference for mean-variance formulations solved using a multiobjective genetic algorithm.

Complications also arise when modeling supply side uncertainty such as link disruptions that may occur from non-recurrent incidents. Link failures that are modeled as independent states tend to be unrealistic. Du and Nicholson (1997) conclude for a simple network that ignoring dependencies between link components can have significant effects on the results. They do not quantify the cost of such an assumption for a realistic sized network. Chen *et al.* (2002) address capacity reliability by modeling continuous link capacities with multivariate normal distributions. They use an orthogonal transform method developed by Chang et al (1994) to convert Monte Carlo simulations of the marginal distributions to that of a joint distribution. However, representing capacities with normal distributions is not ideal because of the bounded nature of capacities. Lo and Tung (2003) argue that the correlated methods are too complex for larger, more realistic networks and instead choose to model the link capacities as independent uniform random distributions. Siu and Lo (2008) extend the model to having both stochastic capacity and demand. Sumalee and Watling (2003, 2008) propose a failure model based

on correlating the causes behind the failures at links instead of at the links themselves. This method is developed because simulating multi-dimensional distributions for all the links in a network is argued to be too complex. Their proposed approach keeps the conditional link degradations independent from each other but still maintains a degree of correlation through the causes, such as the presence of "snow" and "high winds". The problem of multi-dimensional distributions still exists if many causes are considered jointly. Moreover, it these causes are generally exogenous factors that are out of the knowledge domain of transportation planning or operational agents. For example, the causes "snow" and "high winds" may have correlation with each other, but it does not make practical sense for a toll managing agency to model the probability of there being snow on a particular day.

From a problem-centric perspective, all the examples that feature multiobjective network design problems under supply side uncertainty – in this case, the robust network design problem formulated as a mean-variance model – share common research gaps. First, problem sizes are constrained due to the use of multi-objective solution algorithms that are notoriously inefficient when confronted with costly function evaluations. Second, link disruptions at a network level are not easy to model realistically.

The goal of this study is to improve techniques related to those two issues to solve large scale robust network design problems subject to link disruptions. We first propose a new stochastic search metaheuristic for multiobjective problems based on a surrogate model that is shown to converge faster than the multiobjective GA. Second, we propose a novel network degradation simulation model for generating scenarios with correlated link failures. Because of these two contributions, we are able to conduct numerical tests on a classic Sioux Falls network that falls under two different probability regimes representing peak and off-peak seasons. Tests are conducted to measure the gap in ϵ-dominance due to falsely assuming independence in link failures for a given set of correlations, and to quantify the value of flexibility of being able to adapt toll prices in a two-regime setting.

2. LITERATURE REVIEW

2.1. The Robust Toll Pricing Problem

Throughout this paper, the toll pricing problem is discussed as representative of a continuous network design problem. Typically, toll pricing is used as a strategy to reduce congestion or to earn revenue (Small, 1992). Li *et al*. (2007) are one of the first to propose using toll pricing as a strategy to manage uncertainty in a network with stationary stochastic OD demand and link capacity. Boyles *et al*. (2010) propose that toll prices that respond to changes in network capacities can be useful if the information is provided to travelers. The following formulation is from Chen *et al.* (2006) with simulated scenarios representing realized instances.

The following discussion assumes that a set of S scenarios are simulated for which the objective is to maximize expected social welfare and minimize the variance, as shown in the vector in Equation (1). The stochastic element in the proposed model is the capacity of the links in the network as opposed

to the OD demand. The objective of the sub-function for a single scenario is chosen to be social welfare maximization as shown in Equation (4).

$$\max F = (f_1, f_2) \tag{1}$$

subject to

$$y_{max} \geq y_a \geq 0, \qquad a \in \bar{A} \tag{2}$$

$$y_a = 0, \qquad a \in A \backslash \bar{A} \tag{3}$$

where

$$f_1 = E[Z_1(y,\omega)]$$
$$f_2 = -V[Z_1(y,\omega)] = -E[Z_1^2(y,\omega)] + E^2[Z_1(y,\omega)]$$

$$Z_1(y,\omega) = \sum_{rs \in W} \int_0^{d_{rs}(y)} D_{rs}^{-1}(s)ds - \sum_{a \in A} t_a\big(x_a(y), K(\omega)\big)x_a(y) \tag{4}$$

f_1 and f_2 are the first and second moments, respectively, of the stochastic social welfare. W is the set of OD pairs. x_a and d_{rs} are determined by the lower level problem defined in Equations (5) – (8). The lower level problem assumes that travelers behave selfishly according to Wardrop's principle of user equilibrium; that is, they would choose the route from an origin to a destination such that they cannot change their choice without worsening their travel time due to congestion. Because of the presence of congestion, the problem cannot be treated as a single level problem and needs to include the equilibrium constraint as the lower problem.

$$Z_2 = \min \sum_{a \in A} \int_0^{x_a} \left(t_a(v, K_a(\omega)) + \frac{1}{\psi} y_a \right) dv - \sum_{rs \in W} \int_0^{d_{rs}(y)} D_{rs}^{-1}(s)ds \tag{5}$$

s.t.

$$\sum_k f_k^{rs} = d_{rs}, \qquad rs \in W \tag{6}$$

$$x_a = \sum_{rs \in W} \sum_k f_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \tag{7}$$

$$f_k^{rs} \geq 0, \quad \forall k, rs, \qquad x_a \geq 0, \qquad a \in A \tag{8}$$

d_{rs} is the demand function along OD rs defined as shown in Equation (9) for a route cost c_{rs} and parameters d_{rs}^0 and γ, one form of which might be:

$$d_{rs} = d_{rs}^0 \exp(-\gamma c_{rs}) \tag{9}$$

$D_{rs}^{-1}(\,)$ is the inverse of the demand function d_{rs}, where the form corresponding to Equation (9) would be $D_{rs}^{-1}(d_{rs}) = -(1/\gamma)\ln\left(\frac{d_{rs}}{d_{rs}^0}\right)$;

x_a is the flow at link a;
t_a is the travel time across link a defined as a link performance function, one form of which is commonly determined by the Bureau of Public Roads function (BPR) shown in Equation (10) with capacity parameter K_a and free-flow travel time t_a^0 for each link a;

$$t_a = t_a^0\left(1 + 0.15\left(\frac{x_a}{K_a}\right)^4\right), \qquad a \in A \tag{10}$$

y_a is the toll price set on link $a \in A$;
A is the set of links;
$\bar{A}$ is the subset of links that can be priced, where the complement is A\$\bar{A}$;
ψ is the value of time (assumed homogeneous);
f_k^{rs} is the flow along path k for O-D pair $rs \in W$;
$\delta_{a,k}^{rs}$ is the link-path incidence, 1 if link a is on path k for OD pair rs, 0 otherwise;
$\omega \in S$ is one realization of the capacities K_a at each link a, for up to $|S|$ scenarios.

This formulation evaluates a number of simulated scenarios for a given set of probability distributions, maximizing the mean social welfare of all the simulated scenarios while minimizing the variance. A further description of the simulation mechanism for each scenario is described in greater detail in Section 4.

A multiobjective robust network design problem can result in a non-convex Pareto optimal frontier (Sharma *et al*., 2009). Solution methods that use *a priori* articulation of preferences such as minimizing different weighted sums of the objectives do not work well if the Pareto frontier is non-convex (Das and Dennis, 1997). Methods exist to overcome the non-convexity issue, such as the epsilon constraint approach which converts objectives to constraints within some predefined epsilon value. By varying the epsilon value, the Pareto set can be obtained even if it is non-convex. The approach works well for bi-objective problems but not so much for three or more objectives.

In the multiobjective problems and methods reviewed by Marler and Arora (2004), a genetic algorithm (GA) is one of the more conventional methods for solving multiobjective problems with *a posteriori* articulation of preferences. GAs are ideal for multiobjective optimization because they maintain a population of solutions in each generation, which works well for finding approximate Pareto optimal sets. Indeed, this is the method used by Chen *et al.* (2006) and Sharma *et al*. (2009) for solving robust network design problems for small networks using a scenario simulation approach.

Evolutionary algorithms tend to have a heavy computational cost because each generation requires function evaluations to obtain fitness values for each member of the population. This is particularly an issue for robust network optimization problems, which can feature high dimensionality and computationally costly function evaluations. This high cost is the motivation for incorporating computationally cheap surrogate models to speed up convergence (Voutchkov and Keane, 2010). Surrogate models, or metamodels, are defined as computationally cheap models trained to serve in place of the true function, which can be costly to evaluate in many practical applications (on the order of minutes or even hours for a single evaluation, as is the case here).

2.2. Multiobjective Problem Concepts Defined

Given the motivational example of the robust toll pricing problem setup as a multiobjective problem, there needs to be a clear definition of the problem class and notation. The problem definition and notation from Schütze *et al*. (2008) is used. Consider a continuous unconstrained multiobjective optimization problem shown in Equation (11).

$$\min_{x \in \mathbb{R}^n} \{F(x)\} \tag{11}$$

where the function *F* is defined as the vector of *k* objective functions shown in Equation (12):

$$F: \mathbb{R}^n \to \mathbb{R}^k, F(x) = \left(f_1(x), \dots, f_k(x)\right) \tag{12}$$

and where each $f_i: \mathbb{R}^n \to \mathbb{R}$ is continuous.

A solution y is dominated by another solution x if every dimension of F(y) is suboptimal to F(x), or $x \prec y$. A weaker concept of dominance, called ϵ-dominance, is used to evaluate approximate Pareto sets obtained from multiobjective heuristics. From Schütze**,** a solution *x* is said to ϵ-dominate *y* (in short, $x \prec_\epsilon y$) with respect to Equation (11) if $F(x) - \epsilon \leq_p F(y)$ and $F(x) - \epsilon \neq F(y)$, where a vector A $\leq_p$ B if A(i) is less than B(i) for all i. Therefore, a set $F_\epsilon \subset \mathbb{R}^n$ is called an ϵ-approximate Pareto set of the optimization problem in Equation (1) if every point $x \in \mathbb{R}^n$ is ϵ-dominated by at least one $f \in F_\epsilon$, i.e.,

$$\forall x \in \mathbb{R}^n: \exists f \in F_\epsilon: f \prec_\epsilon x$$

From this concept of ϵ-dominance, approximate Pareto sets can be evaluated. The following theorem sets the sufficient conditions for convergence of a stochastic search algorithm that includes an update of the archive in each iteration to an ϵ-approximate Pareto set. The proof is presented in Schütze *et al*. (2008).

Schütze Theorem 3.2. Given a multiobjective problem *F*: $\mathbb{R}^n \to \mathbb{R}^k$, where *F* is continuous, and let $Q \subset \mathbb{R}^n$ be a compact set and $\epsilon \in \mathbb{R}_+^k$. Further let

$$\forall y \in Q, \forall \eta > 0: \; P\left(\exists l \in \mathbb{N}: \mathrm{P}_l \cap B_\eta(y) \cap Q \neq 0\right) = 1 \tag{13}$$

where *Q* is the domain of Equation (11), P_l is a candidate population at the lth iteration, and B_η *n*-dimensional ball of radius η. Then an application of any basic stochastic search and archive update algorithm that compares dominance of every point in P_l with every point in the archive Ω_l leads to a sequence of archives $\Omega_l, l \in \mathbb{N}$, such that there exists an $l' \in \mathbb{N}$ with probability one such that Ω_l is an ϵ-approximate Pareto set for all $l \geq l'$. ▪

2.3. Surrogate Models for Single-Objective Optimization

A number of reviews have been conducted on optimization models using surrogate models (Evan *et al.*, 1991; Simpson *et al.*, 2001; Eldred and Dunlavy, 2006; Wang and Shan, 2007). There are a number of

different types of surrogate models, including polynomial response surfaces, neural networks, kriging, or radial basis functions. Response surfaces have typically been either linear or quadratic polynomial regression models that fit sample data to determine response values in place of using computationally costly functions. Response surface methods were traditionally used as approximation or fitting methods. When used in such a manner to approximate the true function, the results depend on the accuracy of the surrogate model. Furthermore, there is no guarantee of convergence when using surrogate models in place of true functions.

Efforts turned toward the use of surrogate models as tools for optimization instead of for fitting and approximation. Gutmann (2001) proves that convergence can be achieved with the use of radial basis functions (RBFs) as an iterative search algorithm for global optimization. Surrogate models such as RBFs can be used for optimization without as much concern for accuracy if they are used as part of an iterative search algorithm instead of a direct approximation of the true function. Björkman and Holmström (2000) demonstrate the efficiency of the RBF optimization method on a number of test functions. Fang and Horstemeyer (2006) show that RBFs are generally more accurate than polynomial response surface models.

$$s(y) = \sum_{i=1}^{l} \lambda_i \phi(\|y - y_i\|) + p(y), \qquad y \in \mathbb{R}^n \tag{14}$$

where

$\lambda \in \mathbb{R}^l$ is a coefficient of the RBF;
$p(y)$ is a 1st order polynomial with n+1 coefficients μ;

A number of different RBFs can be used to fit $\phi(r; c)$, as shown in Fang and Horstemeyer (2006) and partially represented in Table 1.

Table 1. Commonly used radial basis functions

Name	Function $\phi(r; c)$
Linear	r
Cubic	r^3
Thin-Plate Spline	$r^2 \log(cr), 0 < c \leq 1$
Gaussian	$\exp(-cr^2), 0 < c \leq 1$
Multiquadric	$\sqrt{r^2 + c^2}, 0 < c \leq 1$
Inverse Multiquadric	$\frac{1}{\sqrt{r^2 + c^2}}, 0 < c \leq 1$

The coefficients $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^{n+1}$ of the function $s(y)$ can be obtained by solving the system in Equation (15) (f is the set of evaluated objective values up to n_{cycle} iterations).

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \phi(\|y - y_i\|) & p(y) \\ p(y)^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{15}$$

Regis and Shoemaker (2007) refine the method into a stochastic search framework which they call the metric-stochastic radial basis function (MSRBF) algorithm. They test the RBF-based algorithm's efficiency compared to numerous surrogate models. They also define two variations of the MSRBF: the global MSRBF which searches the whole design space in one run, or the multi-start local MSRBF. The latter algorithm conducts a random search through a stochastic local trust region that decreases in size until local convergence is reached, at which point a new experimental design is re-started. It is shown that the multi-start local MSRBF is significantly more efficient than the global MSRBF for large-dimensional problems. One critique of response surface and radial basis function models is that the computational cost can become burdensome as the size of the training set increases. This is avoided to a large degree in the multi-start local MSRBF, since the training set would be reset when local convergence is achieved.

Chow *et al.* (2010) confirm that the multi-start local MSRBF is a better performing algorithm for the continuous network design problem for transportation networks under congestion, testing both versions of the algorithm against a genetic algorithm on the 416-node Anaheim, CA network with a 31 dimensional design set. The multi-start local MSRBF algorithm from Regis and Shoemaker (2007) is shown in Algorithm 1.

Algorithm 1 (Multi-start local MSRBF, Regis and Shoemaker, 2007)

---

1. Generate initial experimental design.
2. Do costly function evaluations of initial design.
3. While $n < N_{max}$
    a. Fit/Update RBF using Equation (15).
    b. Randomly generate candidate points in local neighborhood of best point using an inverse cumulative normal distribution
    c. Evaluate candidate points with RBF and select approximate best point for next function evaluation.
    d. Set n=n+1. Do costly function evaluation for nth point.
    e. If no improvement for a number of consecutive iterations, reduce size of local neighborhood (reduce standard deviation to half). If maximum number of size reductions reached, reset and go to Step 1, else go to step 3a.
4. Return best solution found.

2.4. Surrogate Models for Multiobjective Optimization

Applying surrogate models to find Pareto optimal sets for multiobjective problems is a relatively new concept. Wilson *et al.* (2001) use kriging and response surface models as substitute for the true function from which a Pareto set is obtained. The surrogate model is not updated and the Pareto set is based purely on the approximate values, which has a strong reliance on the accuracy of the surrogate model. Yang *et al.* (2002) use kriging with a multiobjective genetic algorithm (MOGA). An initial experimental design is generated from which the surrogate model is constructed. A MOGA solution set is then

obtained from the surrogate model. While this model features updating of the surrogate model, it also has a reliance on the accuracy of the model. Farina and Amato (2005) develop a similar strategy using neural networks. Chafekar *et al.* (2005) apply a quadratic response surface model to a MOGA, and incorporate a swapping strategy to mix the population samples over multiple generations. Knowles (2006) extends the Efficient Global Optimization (EGO) model to include kriging to construct a probability function to evaluate new candidates. Rodríguez *et al.* (2009) apply a neural network approach to an evolutionary algorithm, but the model is designed like many of their predecessors to rely on the accuracy of the surrogate as an approximation of the true function. Voutchkov and Keane (2010) provide a review of surrogate models for multiobjective problems, indicating with test functions that they can speed up convergence by orders of magnitude over conventional multiobjective evolutionary algorithms.

Shan and Wang (2005) are the first to apply RBFs to a multiobjective problem, and also consider using the surrogate to accelerate the true function's evaluation. They use RBFs to construct a sample guidance function, analogous to a cumulative distribution function, to generate extreme sample points. All the candidate solutions are evaluated with the true function so that there is no reliance on the accuracy of the surrogate model. However, their search approach is similar to Regis and Shoemaker's global MSRBF, which searches the whole design space, and has been shown to be inferior to the multi-start local MSRBF for larger size problems. Their test function results are not compared with any conventional algorithms either.

Taking the lessons learned from the MSRBF development and the multiobjective surrogate modeling review, a multiobjective algorithm should use surrogates for search instead of as approximation of the true function Pareto set. An algorithm that features a multi-start local trust region type search instead of a global design space search may operate more efficiently. Convergence needs to be proven. These considerations are made in the design of the proposed extension of the multi-start local MSRBF to a multiobjective MSRBF algorithm (MO-RBF).

3. PROPOSED MULTIOBJECTIVE RADIAL BASIS FUNCTION (MO-RBF) METAHEURISTIC

3.1. Algorithm

Several extensions to Algorithm 1 are needed to address multiobjective problems such as the robust toll pricing problem. First, the proposed MO-RBF method is used to construct a set of candidate points by simultaneously interpolating each of the K objectives.

The single objective multi-start MSRBF algorithm resets with a new initial sample set when local convergence is reached. For the MO-RBF algorithm, this approach is modified so that a reset would still occur, but the lth Pareto set would be stored in an archive Ω_l.

In multiobjective problems it is desirable to have Pareto sets that spread out over a range instead of clumping together. A distance criterion is used to evaluate how far new solutions are from the

current Pareto set. The distance criterion for the MO-RBF method for choosing the optimal candidate point is obtained by the maximin distance from the existing approximate Pareto set's objective vectors instead of selecting the furthest point from the existing local MSRBF solutions.

Lastly, the choice of the best fitting solution to randomly generate the neighborhood of new candidate points is determined by randomly assigning a point in the current Pareto set to generate each new candidate. This differs from the MSRBF approach, which uses a best fit solution as well as a distance criterion. The input parameters are shown below and the algorithm is presented in Algorithm 2.

INPUT PARAMETERS:

$l_{\max}$ = maximum number of function evaluations;
P = number of candidate points;
l_0 = initial set of feasible solutions generated via Latin hypercube sampling;
σ = initial vector of standard deviations for searching the variable space using an inverse Normal random draw;
r_{max} = maximum number of consecutive searches without success for a given σ;
ζ_{max} = maximum number of $\sigma/2$ reductions before the algorithm is re-started;
$\phi(r;c)$ = choice of radial basis function and parameter c for evaluating candidate points, as shown in Table 1.

Algorithm 2: Proposed MO-RBF

---

0. Initiate an empty Pareto set Ω_0. Initiate r_{max} and ζ_{max}. Initiate l = 0 and n_{cycle} = 0.
1. Initiate $\sigma \in \mathbb{R}_+^n$. Generate l_0 initial solution samples. $l := l + l_0$. Let $l_{cycle} = l_0$. Set ζ = 0 and r = 0.
2. Evaluate the designs with the objective function vector.
3. Set Ω_l as the non-dominated set from union of Ω_{l-1} and initial design objective vectors.
4. Fit/update Equation (14) to obtain interpolated value s_k by solving Equation (15) for each k of the K objectives.
5. Generate P candidate points: $x_p := \bar{x}_p - \sigma^D \Phi^{-1}(q)$, where $p \in P$, Φ^{-1} is the inverse standard normal, q is a randomly generate percentile, σ^D is a diagonal matrix, and $\bar{x}_p$ is a uniformly selected point in Ω_l. Evaluate each candidate point x_p with Equation (14) for the K objectives $s_k(x_p)$.
6. Construct a temporary non-dominated set φ_p from candidate points, select point that satisfies Equation (16):

$$y_{n+1} = \operatorname{argmax}\left[\min_{y_i \in \Omega_l}\left(\left\|s(x_p) - f(y_i)\right\|\right) : x_p \in \varphi_p\right] \tag{16}$$

7. Let n = n+1. Evaluate the objective functions as described in steps 3 – 4 to obtain $F(y_{n+1})$.
8. $l := l + 1$. $\Omega_{l+1} := \Omega_l$. If $\exists a_i \in \Omega_{l+1}: a_i \prec_\epsilon y_{n+1}$, then go to Step 9, else

a. For each solution $a_i \in \Omega_{l+1}$, if $y_{n+1} \prec_\epsilon a_i$ then $\Omega_{l+1} := \Omega_{l+1} \backslash \{a_i\}$.

b. $\Omega_{l+1} := \Omega_{l+1} \cup y_{n+1}$

9. If $\Omega_{l+1} = \Omega_l$, $\zeta := \zeta + 1$. If $\zeta > \zeta_{max}$, then $\sigma := \sigma/2$, $\zeta := 0$ and $r := r + 1$. If l = L, where L is maximum number of iterations, end. If $r \leq r_{max}$ and $l \leq l_{\max}$, go to Step 5, else go to Step 1.

The MO-RBF algorithm requires that the problem be continuous, the domain *Q* be a compact set, and include a candidate selection in Step 6 and an archive update in Step 8 based on the update algorithm in Schütze et al. (2008). Steps 5 – 7 generate random candidate points and select one as the most distant from every other point in the current Pareto set Ω_l. This algorithm differs from existing surrogate-based multiobjective algorithms in a number of ways. In this algorithm, a multi-start local search is used based on evidence indicating a better performance in single objective cases. Instead of having to sort a Pareto set to obtain a sample guidance function, the analogy of local random search using an inverse normal distribution is extended to the Pareto set to better handle a large number of *K* objectives. The Pareto set is maintained separately and requires true function evaluation to include a candidate in the set. The proposed algorithm also prevents having a large training set size to maintain a manageable computational speed.

3.2. Proof of Convergence

We show that our algorithm falls under the class of stochastic search and update algorithms described by Schütze et al. (2008), and then rely on Schütze Theorem 3.2 for stochastic convergence.

First, suppose the RBF interpolation function always results in selecting the worst fitting candidate point among the set of candidate points generated randomly from a Gaussian distribution over a compact space. Nonetheless, since the number of candidate points being drawn randomly is finite, there exists a probability greater than zero that all the candidate points will improve the Pareto set and thus even the worst performing RBF interpolation function has a probability of improving the Pareto set so that Ω_{l+1} is updated. Since $p(\Omega_{l+1} \prec_e \Omega_l) > 0$, then there exists a number of iterations *n* such that $\Omega_{l+n} \prec_e \Omega_l$. If we replace a single iteration with *n*, then the conditions for Schütze's Theorem 3.2 are satisfied. ▪

Since Schütze's Theorem 3.2 is defined over a compact space, then the theorem naturally extends to constrained multiobjective problems with continuous variable domain where the constrained space would also be compact. Therefore the convergence theorem applies to the proposed algorithm for constrained multiobjective problems with continuous variables.

While the algorithm will stochastically converge, the speed of convergence can depend upon the choice of such parameters as the size of $\sigma \in \mathbb{R}_+^n$ in Step 1 of Algorithm 2. Since the algorithm restarts the size after a number of iterations, it doesn't affect the proof of convergence, but it can result in slow speeds for some problems.

Like the MSRBF algorithm for single objectives, Schütze's theorem (and therefore this algorithm) relies on the multiobjective problem having only continuous decision variables in order to guarantee stochastic convergence. This is an issue that exists with the other surrogate-based optimization models as well, and is addressed when combined with evolutionary algorithms. Nevertheless, a large number of problems such as the class of large-scale multiobjective continuous network design problems can be

solved using this heuristic alone. Future work will examine the combination of this algorithm with a genetic algorithm to tackle discrete decision variables.

3.3. Convergence Performance Test

The speed of the MO-RBF is empirically tested on two types of problems: a set of known test problems with known Pareto optimal solutions in this section, and relatively compared to a commercial NSGA solver in Matlab for the robust toll pricing problem in Section 5.

MO-RBF is numerically shown to significantly outperform some conventional heuristics in the convergence measure, as defined in Deb *et al*. (2002), for multiple test problems. These tests are necessary to provide some benchmarks so further extensions of the algorithm for transportation network design problems can be conducted and compared to these results. Note that none of the other surrogate-based models cited in Section 2 have been compared to the 10- to 30-dimensional ZDT test problems defined by Ziztler (2000) using the convergence and diversity measures provided in Deb *et al.* (2002). These are absolute measures used to define the deviation of an approximate Pareto set from the true Pareto set.

When the true Pareto set is known, absolute measures are more effective than relative measures of dominance. For example, consider a true Pareto frontier along the line defined in bi-objective space from (0,1000) to (1000,0). A first algorithm might have 11 points allocated 1 unit above the line, spread evenly between (0, 1001) to (500, 501). A second algorithm might have one point located at (0, 1000), and ten points scattered at (900, 500), (901, 499), (902, 498)… (909, 491). It is clear that the solution from the second algorithm weakly dominates the solution from the first algorithm, but it should also be clear that the first algorithm is closer in absolute terms to the true solution.

In most cases the true solution is unknown, but Deb *et al*. (2002) collect a number of problems with known optimal solutions (designed to fall along a single dimension) to examine the absolute performance of their NSGA-II algorithm against other methods, as shown in their Table 2. A detailed overview of the test for one problem is described, followed by test results for that problem as well as for three other test problems in that paper. The ZDT series of test problems are defined such that even for a large number of variables the optimal solution can be clearly depicted along one dimension. This makes it very convenient to evaluate ϵ-approximate Pareto sets against the true Pareto optimal set. These problems, which were specially designed so that they would be especially difficult to solve using evolutionary algorithms, were first presented in Zitzler, Deb and Thiele (2000), and then presented again in Deb *et al.* (2002), but with slightly different notation.

Test problem ZDT6 has a Pareto set that is known to be both non-convex and non-uniformly distributed. Its similar properties to Pareto sets for multiobjective network design problems make it an appealing test problem to compare the algorithms' effectiveness. The problem is to minimize f_1 and f_2, where $x \in \mathbb{R}^{10}$ with box constraints $x \in [0,1]$. The true Pareto optimal set lies along $x_1 \in [0,1], x_i = 0$ for i = 2, 3, … , 10. Deb *et al.* (2002) provide two performance measures that compare an approximate

Pareto set with a true Pareto set for these ZDT functions: a convergence criterion and a diversity criterion. The convergence criterion is the average distance between each solution and its closest point in the true Pareto optimal set. The diversity criterion is the average of the absolute value of the deviations from the mean. The two criteria are defined in Equations (17) – (18).

Convergence Criterion:

$$\Upsilon = \frac{\sum_{i=1}^{|\Omega_L|}\left[\min_{\mathrm{h}}\|F_h - \Omega_i\|\ \right]}{|\Omega_L|} \tag{17}$$

where F_h is one of a set of known Pareto optimal solution points selected from a set H of uniformly distributed points and Ω_i is the ith approximate Pareto optimal value obtained from the algorithm.

Diversity Criterion:

$$\Delta = \frac{d_{f1} + d_{f2} + \sum_{i=1}^{|\Omega_L|-1}\left|d_i - \bar{d}\right|}{d_{f1} + d_{f2} + (|\Omega_L| - 1)\bar{d}} \tag{18}$$

where d_{f1} and d_{f2} are the Euclidean distances between the extreme solutions and the boundary solutions of the non-dominated set, d_i is the distance between the ith and the $(i+1)^{th}$ solution, and $\bar{d}$ is the average of all distances.

The mean and variance values are obtained from 10 runs, each involving 25,000 function evaluations. The MO-RBF is compared against other heuristics presented in Deb *et al.* (2002), including the non-sorting genetic algorithm (NSGA-II), Zitzler and Thiele's SPEA, and Knowles and Corne's PAES. According to Rodríguez *et al.* (2009), these conventional evolutionary algorithms tend to require up to 20,000 fitness function evaluations to converge.

3.4. Convergence Test Results

The MO-RBF reaches the same order of magnitude of the convergence measure after 1,250 iterations, or 5 percent of the 25,000 iterations. By 6,250 iterations, the convergence measure is better than all four of the other algorithms from Deb *et al.* (2002). This is in agreement with the relative convergence speed observed by Voutchkov and Keane (2010). Unless noted otherwise, the MO-RBF parameters are set to l_0 = 21, $\sigma = 0.1*(Y_{max} - Y_{min})$, r_{max} = 5, ζ_{max} = 10, P = 1000, and $\phi(r;1) = r^2 \log r$ (thin plate spline). Table 2 summarizes the results from Deb *et al*. for ZDT6 with the MO-RBF results appended.

Table 2. Comparison of MO-RBF (10 runs) with other Algorithms for ZDT6 from Deb *et al.* (2002)

Algorithm	Mean Convergence	Variance Convergence	Mean Diversity	Variance Diversity
NSGA-II (real)*	0.296564	0.013135	0.668025	0.009923
NSGA-II (binary)*	7.806798	0.001667	0.644477	0.035042

SPEA*	0.221138	0.000449	0.849389	0.002713
PAES*	0.085469	0.006664	1.153052	0.003916
MO-RBF (250 iter)	2.118297	2.928835	0.943079	0.020213
MO-RBF (1,250 iter)	0.282318	0.079473	1.068302	0.084596
MO-RBF (2,500 iter)	0.100420	0.005766	1.044337	0.109991
MO-RBF (6,250 iter)	0.046776	3.76×10^{-4}	1.009616	0.082729
MO-RBF (12,500 iter)	0.022773	1.50×10^{-4}	0.970562	0.052642
MO-RBF (18,750 iter)	0.016211	5.85×10^{-5}	0.924199	0.036671
MO-RBF (25,000 iter)	0.011872	2.19×10^{-5}	0.895763	0.024061

*Values reported in Deb *et al*. (2002) for 25,000 iterations.

The results clearly show that MO-RBF has an order of magnitude better convergence rate than any of the other algorithms presented in Deb *et al*. (2002) for ZDT6. Figure 1 shows a comparison of the results of the first run with the NSGA-II and SPEA obtained from Deb *et al.* (2002). At 25,000 iterations, MO-RBF has a large number of points like NSGAII, but whereas NSGAII is trapped at a local front compared to SPEA, the MO-RBF lies on the true Pareto optimal front.

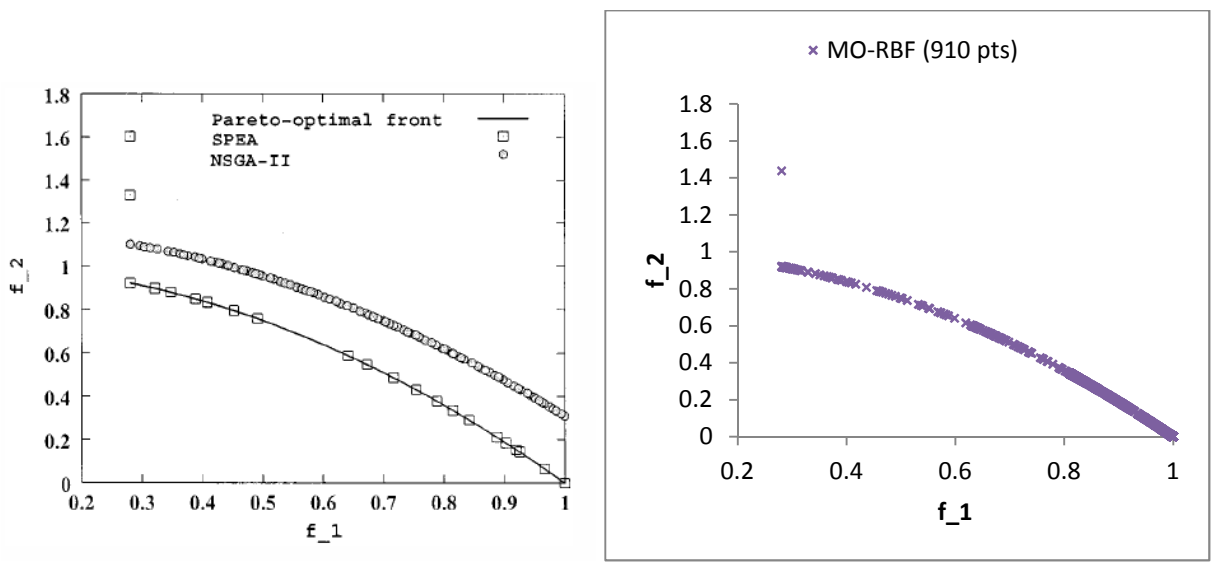

Figure 1. Comparison of MO-RBF decision space from a sample run (right) to Deb et al (2002)'s results (left)[1].

This comparison is conducted for ZDT1, ZDT2, and ZDT3 as well. ZDT1 has a convex Pareto-optimal front and the solutions are uniformly distributed in the search space. Test function ZDT2 is the non-convex counterpart to ZDT1, while the Pareto-optimal front in ZDT3 consists of several non-contiguous convex parts; its search space is unbiased. The results are compared to the existing algorithms using the same measures and results provided from Deb *et al.* (2002). The convergence measure results are shown in Table 3 while the diversity measures are shown in Table 4.

[1] This figure is used with the permission of the authors.

Table 3. Mean (First Rows) and Variance (Second Rows) of Convergence Measure Comparison for ZDT1 – ZDT3 from Deb *et al.* (2002) with MO-RBF (10 runs each)

Algorithm	ZDT1	ZDT2	ZDT3**
NSGA-II (real)*	0.033482	0.072391	0.114500
	0.004750	0.031689	0.007940
NSGA-II (binary)*	0.000894	0.000824	0.043411
	0	0	0.000042
SPEA*	0.001799	0.001339	0.047517
	0.000001	0	0.000047
PAES*	0.082085	0.126276	0.023872
	0.008679	0.036877	0.00001
MO-RBF (250 iter)	0.240677	0.129545	1.38196
	0.007903	0.002107	0.183661
MO-RBF (1,250 iter)	0.026800	0.025885	1.219511
	2.75×10^{-5}	6.39×10^{-5}	0.130703
MO-RBF (2,500 iter)	0.018145	0.020994	0.751587
	1.55×10^{-5}	3.09×10^{-5}	0.090882
MO-RBF (6,250 iter)	0.013602	0.017244	0.285260
	1.20×10^{-5}	1.79×10^{-5}	0.040123
MO-RBF (12,500 iter)	0.010660	0.014899	0.068150
	5.36×10^{-6}	1.19×10^{-5}	0.000990
MO-RBF (18,750 iter)	0.009459	0.013457	0.043548
	4.37×10^{-6}	1.31×10^{-5}	0.000121
MO-RBF (25,000 iter)	0.008737	0.014272	0.040203
	3.69×10^{-6}	4.25×10^{-6}	0.000150

*Values reported in Deb *et al*. (2002) for 25,000 iterations.

** Due to computational cost from a large training set, the MO-RBF runs for this function are capped at 2,000 iterations per local run. Performance does not appear to be compromised.

Table 4. Mean (First Rows) and Variance (Second Rows) of Diversity Measure Comparison for ZDT1 – ZDT3 from Deb *et al.* (2002) with MO-RBF (10 runs each)

Algorithm	ZDT1	ZDT2	ZDT3
NSGA-II (real)*	0.390307	0.430776	0.738540
	0.001876	0.004721	0.019706
NSGA-II (binary)*	0.463292	0.435112	0.575606
	0.041622	0.024607	0.005078
SPEA*	0.784525	0.755148	0.672938
	0.004440	0.004521	0.003587
PAES*	1.229794	1.165942	0.789920
	0.004839	0.007682	0.001653
MO-RBF (250 iter)	0.806701	0.763158	0.698000

	0.021323	0.023904	0.008826
MO-RBF (1,250 iter)	0.734811	0.687225	0.869491
	0.003862	0.015946	0.016198
MO-RBF (2,500 iter)	0.711259	0.700923	0.858302
	0.002027	0.013710	0.016715
MO-RBF (6,250 iter)	0.659676	0.690858	0.967296
	0.001961	0.013219	0.020099
MO-RBF (12,500 iter)	0.617549	0.680229	0.834877
	0.001591	0.013937	0.010559
MO-RBF (18,750 iter)	0.608900	0.664425	0.824814
	0.000574	0.016056	0.003362
MO-RBF (25,000 iter)	0.585721	0.634339	0.750609
	0.000328	0.002556	0.003742

*Values reported in Deb *et al*. (2002) for 25,000 iterations.

For ZDT1, ZDT2, and ZDT6, the MO-RBF reaches the same order of magnitude convergence measure as the evolutionary algorithms in between 1,250 to 2,500 iterations. For ZDT3, the MO-RBF reaches the comparative range within 12,500 iterations. Obviously the MO-RBF can be further improved by specifying the design parameters for each test function. Certain functions require careful selection of the RBF. For example the thin plate spline did not appear suitable for problem ZDT4, which would need further testing along with other test functions for different types of surrogate models. Overall, these results confirm that the MO-RBF algorithm can be a much faster converging heuristic with applicability to solving complex, large-scale multiobjective problems. In addition, the diversity of the solutions of the four test problems using MO-RBF is in the same range as the evolutionary algorithms. However, these tests confirm that care needs to be placed in the experimental design, and the algorithm may not work well for every problem.

3.5. Approximate Pareto Set Evaluation

The criteria introduced with Equations (17) – (18) do not work for ϵ-approximate Pareto sets because the true Pareto optimal set is not known. For the numerical tests in Section 4, a binary measure needs to be used for quantifying the difference between two approximate Pareto optimal sets.

The "ϵ-indicator" is a binary performance measure defined by Zitzler *et al*. (2003) as an effective quantitative measure to compare two approximate Pareto sets. First, an objective value $z^1 = (z_1^1, z_2^1, \dots, z_n^1) \in Z$ is defined to ϵ-dominate another objective vector $z^2 = (z_1^2, z_2^2, \dots, z_n^2) \in Z$, i.e., $z^1 \succcurlyeq_\epsilon z^2$, if and only if $\forall 1 \le i \le n: z_i^1 \le \epsilon z_i^2$ for a given $\epsilon > 0$. The ϵ-indicator $I_\epsilon(A, B)$ for two approximate Pareto sets *A, B* is defined as the minimum factor ϵ such that any objective vector in *B* is ϵ-dominated by at least one objective vector in *A*. In other words, if $z^1 \succ z^2$ then these exists $\epsilon < 1$ such that z^1 ϵ-dominates z^2, for $z^1 \in A$ and $z^2 \in B$.

$$I_\epsilon(A,B) = \max_{z^2 \in B} \min_{z^1 \in A} \max_{1 \le i \le n} \frac{z_i^1}{z_i^2} \tag{19}$$

While there are certainly other measures of approximate Pareto sets, the "ϵ-indicator" is sufficient for this particular case study where the numerical test problem is bi-objective.

4. PROPOSED NETWORK DEGRADATION SIMULATION MODEL

The second issue relevant to large scale robust toll pricing problem with network disruptions is finding a simple but realistic way of generating disruption scenarios. As discussed, independent link failures are not desired despite the simplicity of implementation. On the other hand, prior efforts in considering link dependencies have either resulted in methods that are deemed unrealistic in other aspects or impractical.

A practical compromise is sought between the method introduced by Chen et al (2002) and by Lo and Tung (2003). In the insurance industry, it is common to model a portfolio of risks with a multivariate Bernoulli random variable to indicate that claims have occurred, and to model the claim amounts as positive random variables (Frostig, 2001). However, even multivariate Bernoulli random variables can quickly escalate in complexity with the size of the dimension (Teugels, 1990).

Curtis *et al*. (2006) overcome this problem of simulating multi-dimensional Bernoulli variables with a method that transforms multivariate Bernoulli random variables into normal distributions. They show that if the mean and variance of the Bernoulli variables are treated as equivalent normally distributed parameters, there exists a threshold value τ such that $\Pr(X>\tau) = \Pr(Z=1)$, where Z is the Bernoulli variable and X is the transformed normal distributed variable with the same mean and variance. The τ can be obtained with Equation (20).

$$\tau = \mu + \sigma\sqrt{2}\,\mathrm{inverf}(1-2\mu) \tag{20}$$

Here the μ and σ are the mean and standard deviation of a Bernoulli variable and inverf is the inverse error function. By converting the covariance matrix to an equivalent matrix for the normal distribution, random samples can then be drawn. In the equivalent covariance matrix, the diagonals are kept the same, and the non-diagonals are determined from a correlation coefficient ρ_0 such that the relationship in Equation (21) exists for each *i,j* pair.

$$\Pr\left(Y_i > \tau_i, Y_j > \tau_j; \rho_0\right) = \Sigma_{\mathrm{ij}} + \mu_{\mathrm{i}}\mu_{\mathrm{j}} \tag{21}$$

To model the stochastic capacity scenarios, the parameter K_a in Equation (10) is treated as a random variable that is uniformly distributed between $[0, \overline{K}_a]$ and conditioned on a *n*-dimensional Bernoulli variable θ. Degradation is defined as the probability that a binary failure occurs ($\theta_a = 1$) followed by uniformly random degree of degradation. $\overline{\mathrm{K}}_{\mathrm{a}}$ is the non-degraded capacity value and n is the number of links in the network. The conditional probability of degree of degradation for a given link is shown in Equation (22).

$$Pr(K_a \leq x | \theta_a = 1) = \frac{x}{\overline{K}_a} \tag{22}$$

Because of the computational expense imposed by the number of scenarios, an efficient sampling method called Latin hypercube sampling (LHS) is used to generate the random seeds in a similar fashion to Chen *et al*. (2006). LHS is a stratified random sampling technique that breaks down the sampling distribution into multiple regions to ensure full coverage of the range of the distribution in the most efficient manner. In particular, the LHS Matlab utility developed by Budiman (2004) is used for convenience.

Monte Carlo simulation is used to generate the marginal standard normal random variables for each scenario $\omega \in S$. Equation (23) is used to perform an orthogonal transform to convert the marginal normal distributions into a jointly distributed multivariate normal distribution. Equation (24) converts the multivariate normal distributed vector to an equivalent multivariate Bernoulli vector.

$$X(\omega) = \Phi_{0,1}\left(V\Lambda^{1/2}\varepsilon_M(\omega)\right) \tag{23}$$

$$\tilde{\theta}_a = 1 \text{ if } \mu_a \geq X_a(\omega), \tag{24}$$

where

$\varepsilon_M \in \mathbb{R}^S$ is a sample $\omega \in S$ of a standard normally distributed random variable;
V is the eigenvector of the standard normal correlation matrix ρ_0;
Λ is the diagonal matrix of eigenvalues from the standard normal correlation matrix ρ_0;
$X \in \mathbb{R}^M$ is the vector of correlated percentiles;
μ_a is the probability of degradation at link $a \in A$;
$\Phi_{0,1}$ is the cumulative distribution function of a standard normally distributed variable;
$\tilde{\theta}_a$ is the ath element of the simulated M-dimensional Bernoulli variable.

The bivariate normal distribution is solved with numerical integration via the *mvncdf* function in MATLAB R2007b. A secant method bounded by [-0.9999,0.9999] is used to find the root value of ρ_0 for Equation (21).

As discussed in Chang *et al*. (1994), there may be errors introduced in this method for estimating the joint distribution using the orthogonal transform method, but it provides a simplified scheme of accounting for dependencies between links in the scenario generation. The simulation method is shown in Algorithm 3.

Algorithm 3: Proposed Multivariate Bernoulli Network Degradation Simulation

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1. Generate *S* sets of marginal standard normal variables using Latin Hypercube sampling.
2. For the desired multivariate Bernoulli mean and covariance matrix, an equivalent set of normal distribution parameters can be obtained with Equations (20) – (21).
3. Apply the orthogonal transform method in Equation (23) to the samples in Step 1 with the parameters from Step 2 to obtain a correlated vector of percentiles for each sample in *S*.

4. Convert each vector back to a multivariate Bernoulli variable using Equation (24) for each sample in *S*.
5. Simulate link degradation with Equation (22).

To ensure that the approximation error is minimal, a comparison of the simulation results for a 76-link network is conducted for different sample sizes using the LHS method as well as a simple random sampling (SRS) method. A correlation coefficient of 0.25 is used for the same pairs of links as in the Sioux Falls network described in Section 5.1. The root mean squared errors (RMSE) are shown in Table 5.

Table 5. Comparison of Capacity Failure Simulation for different sample sizes and sampling methods

	RMSE			
	SRS μ	LHS μ	SRS ρ	LHS ρ
S=50	0.0545	0.0325	0.1452	0.1427
S=100	0.0353	0.0233	0.1061	0.1051
S=200	0.0224	0.0143	0.0741	0.0741
S=1000	0.0106	0.0067	0.0401	0.0400

The RMSEs of the mean values of the Bernoulli distributions (SRS μ, LHS μ) fall within 3 percent using the LHS method for a sample size of 100. The RMSE of the correlation coefficients of the Bernoulli variables (SRS ρ, LHS ρ) are approximately 10 percent for 100 samples, which makes sense since there are an order of magnitude more correlation coefficients to simulate. These results favor the LHS approach for computational efficiency, and suggest using S = 200 scenarios or greater for a RMSE of the μ of 2% or less.

5. NUMERICAL TEST

5.1. Sioux Falls Base Conditions

A numerical test is conducted with the Sioux Falls network commonly used in the network design literature as discussed in Chow *et al.* (2010), and shown in Figure 2. The following changes are made. Ten links that are typically investment candidates in the network design problem are changed to be toll links instead. The value-of-time parameter is set to $\psi = \$10$/hour. For the parameters in Equation (20), $\gamma = 1$ for convenience and the base demand is set to the fixed OD demand values from the Sioux Falls network in the literature.

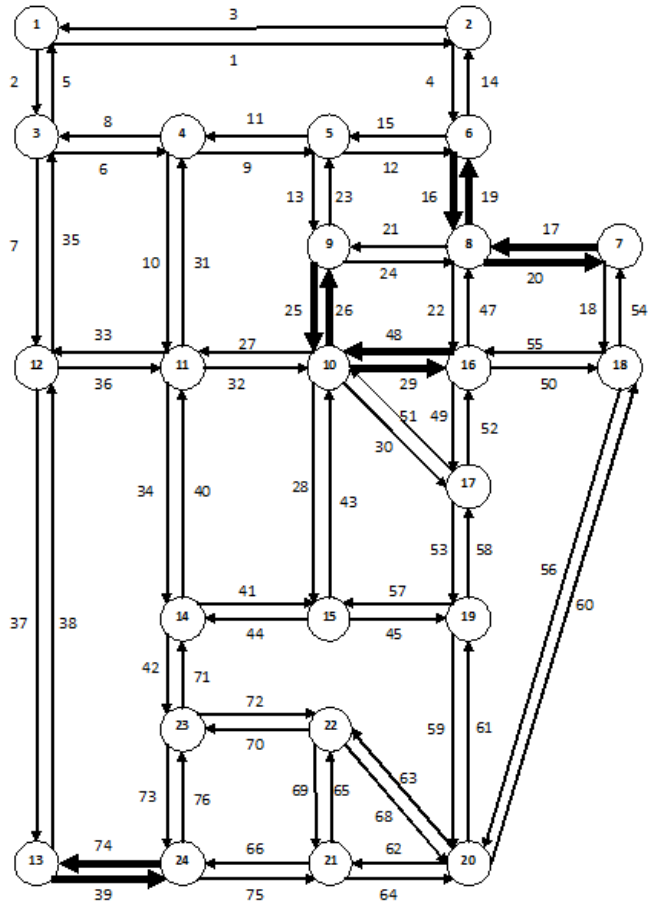

Figure 2. Sioux Falls, SD Network

To establish a baseline, the elastic demand user equilibrium without toll prices is obtained as well as the system optimal equilibrium to represent the first-best marginal toll pricing condition. In addition to the deterministic scenarios, baseline stochastic scenarios are simulated for three hypothetical regimes: a low probability regime (5% link failure probability), a high probability failure regime (50% link failure probability), and a low probability (5%) regime with correlated link failures (50% correlations across all link pairs). The expected user equilibrium and variance under 1000 simulated failure scenarios are obtained for the three stochastic scenarios.

In first-best marginal cost toll pricing, the difference between the marginal travel time and the average travel time at the system optimal (SO) solution represents the optimal toll. A maximum toll of $10 is assumed. The result of the SO solution is summarized along with the optimal social welfare from second-best toll pricing (with only the 10 links allowed for tolling) in Table 6.

The multi-start MSRBF algorithm proposed by Regis and Shoemaker (2007) and applied in Chow *et al.* (2010) to a similar network example is used to obtain the optimal tolls. Parameters are set to n_0 = 21, ρ_{M0} = 0.1*Y_{max}, r_{max} = 5, and ζ_{max} = 10, P = 1000, l_{max} = 2500, and $\phi(r;1) = r^2 \log r$. For the elastic

demand user equilibrium (UE), the Frank-Wolfe algorithm (Sheffi, 1985) is run up to a maximum of 100 iterations with a relative objective tolerance of 0.01.

Table 6. Sioux Falls Baseline Social Welfare Values

DETERMINISTIC	Social Welfare (kveh*hr)	
No Toll User Equilibrium	334.998	
No Toll System Optimum	342.208	
Second-Best Toll User Equilibrium	335.440	
STOCHASTIC (1000 Scenarios)	**Expected Social Welfare (kveh*hr)**	**Variance of Social Welfare (kveh*hr)^2**
No Toll, 5% Independent Link Failure Probability	330.983	11.827
No Toll, 5% Link Failure with +50% Correlations	330.179	269.233
No Toll, 50% Independent Link Failure Probability	284.673	167.872

Three observations can be made from these base values. First, the deterministic social welfare bounded by the first best SO solution does not appear to be significantly higher than the UE solution for this network, and second-best toll pricing of the 10 chosen links appears to have negligible benefits. A transportation agency considering these 10 links under a deterministic setting would dismiss them as potential locations for pricing, and in fact may not even consider toll pricing for this network to be a viable strategy at all, given the limited upper bound from the system optimal solution. Nevertheless, it is shown in the following sections that these same 10 links can be priced effectively as a robust strategy.

Second, when the links have 50% correlation with one another, it results in similar expected social welfare. However, it also leads to a significant increase in the variance of the social welfare, even greater than having independent 50% failure rates.

Third, switching from a low probability failure regime to a high probability failure regime (e.g., from a low hurricane season to a high one) can result in significant shifts in the no toll social welfare measure. The mean social welfare drops by approximately 14 percent, while the variance increases fourteen times.

This section is used to demonstrate four results: 1) the efficiency of the stochastic search algorithm combined with a given number of simulated scenarios; 2) the improvement in the ϵ-approximate Pareto set when compared against a commercial MOGA; 3) the gap in ϵ-dominance due to falsely assuming independence in link failures for a given set of correlations; and 4) the value of flexibility of adapting toll prices in a two-regime setting.

5.2. Efficiency Test under 5% Independent Failure Regime

The stochastic convergence of the simulation method in terms of the number of scenarios is tested for the Sioux Falls network with S = 350 and l_{max} = 300. The MO-RBF algorithm is applied to the 5% independent Bernoulli probability failure regime, using P = 1000 candidate points per iteration. The Pareto sets of five independent runs are shown in Figure 3. The computations were performed on Matlab 7.7.0 (R2008b) on a Windows XP Professional 2002 SP3, Intel Core2 Quad CPU with Q6600 @2.40GHz and 2GB RAM.

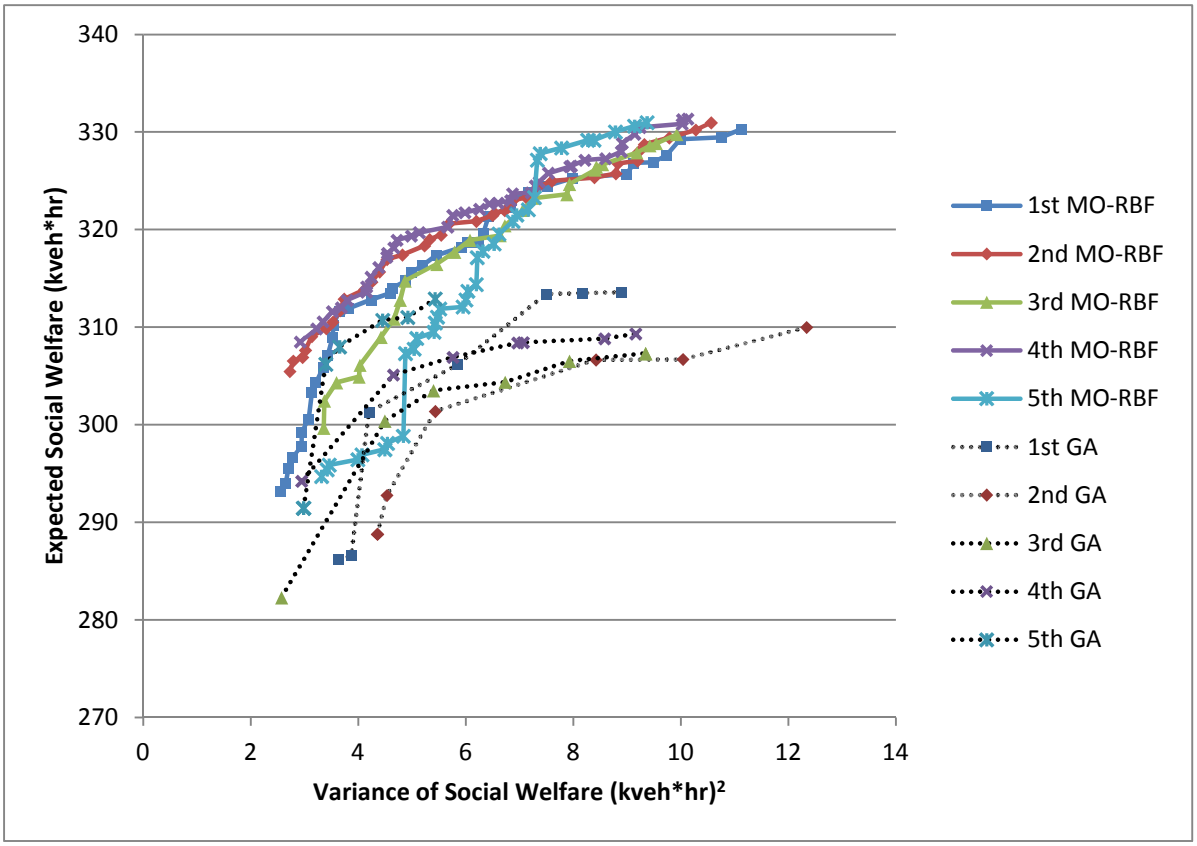

Figure 3. 5% failure regime solution space for different number of scenarios *S* and iterations *L*.

The ϵ-indicators are determined for the first MO-RBF Pareto set compared to the other four Pareto sets. The five ϵ-indicators $I_\epsilon(\Omega_1, \Omega_i)$ = {1.0333, 1.0065, 1.0309, 1.0146} for i = 2 to 5. The variance of the ϵ-indicators is 1.67×10^{-4}, while the standard error of the measure is 1.29%. This error suggests that at 350 scenarios and 300 MO-RBF iterations it is possible to get an ϵ-approximate Pareto set with a standard error in terms of ϵ -dominance by less than 1.5%.

Although Table 6 indicates that setting tolls on the 10 links would only be marginally beneficial from the perspective of maximizing deterministic social welfare, the results shown here in Figure 3 demonstrate that robust toll pricing can be a viable risk management tool, as argued by Li *et al.* (2007) – variance in social welfare can be controlled from 2 to 12 $(\text{kveh*hr})^2$ and expected social welfare from 295 to 333 kveh*hr.

5.3. ϵ-Dominance Test against Multiobjective Genetic Algorithm

Contrary to similar robust network design studies conducted by Chen *et al.* (2006) and Sharma *et al*. (2009), the MO-RBF algorithm allows larger network instances to be tested due to a smaller number of required function evaluations. The Pareto optimal sets obtained using MO-RBF are compared against the commercial *gamultiobj.m* Matlab function, which uses a controlled elitist genetic algorithm, a variant of Deb's (2001) NSGAII.

Five approximate Pareto sets are obtained from the commercial GA Matlab function, and are plotted in Figure 3 as dashed lines alongside the MO-RBF Pareto sets. Default Matlab parameters are used, except for the following: the program ends after 14 generations (300 function evaluations), a population size of 20, a crossover fraction of 0.5, and the same LHS approach is used to generate the initial populations. The same set of simulated scenarios in each of the five MO-RBF runs are re-used in the GA runs to take the stochastic nature of the scenario simulation out of the picture. Graphically the MO-RBF Pareto sets tend to have more dispersed, numerous points that are more dominant than the GA Pareto sets.

The ϵ-indicator is used to quantitatively measure how much the MO-RBF Pareto sets need to be increased to fully dominate the GA Pareto sets, and vice versa. First, a comparison is made of the MO-RBF sets against the GA sets.

$$I_\epsilon(\Omega_{MO-RBF}, \Omega_{GA}) = \{0.9680, 0.9495, 1.3047, 0.9914, 1.1122\}$$

Out of the five runs, the MO-RBF Pareto sets ϵ-dominate the GA sets in three cases ($I_\epsilon(\Omega_{MO-RBF}, \Omega_{GA}) < 1$). On average the MO-RBF set needs to be unilaterally improved by 6.5% to dominate the corresponding GA Pareto set. On the other hand:

$$I_\epsilon(\Omega_{GA}, \Omega_{MO-RBF}) = \{1.4269, 1.5971, 1.0844, 1.0712, 1.0577\}$$

This result shows that none of the five GA Pareto sets ϵ-dominate the MO-RBF sets. On average the GA Pareto set needs to be unilaterally improved by 24.7% to ϵ –dominate the corresponding MO-RBF Pareto set.

5.4. Quantifying the Gap from Falsely Assuming Independent Link Failures

Although Li *et al*. (2007) are one of the first to propose using toll pricing as a strategy to manage uncertainty in a network, their approach treats link degradation risks as independent of one another. Lo and Tung (2003) and Boyles *et al*. (2010) also assume independent link failure probabilities, modeled as independent Bernoulli variables. While Du and Nicholson (1997) suggest that there is a cost to assuming independence, they only demonstrate this for a small network with two nodes and two links and in terms of a single objective measure. This section provides a quantification of the cost of such an assumption for the Sioux Falls network for a multiobjective measure, given actual correlation of 50% between each link when the marginal probability of failure at each link is 5%.

The correlated setting has the same mean probabilities of failure for each link as in the independent case in Section 5.2. However, due to the correlations, the link failures occur in tandem more often. The 50% positive correlation between every link in the network can be interpreted as the effect of link failures due to large scale incidents such as earthquakes or major storms.

One run of the MO-RBF is conducted with S = 350 scenarios and plotted in Figure 4. While the expected social welfare of the Pareto set remains in approximately the same range (280 – 330 kveh-hr), the variance increases significantly just as is the case in Table 6 with no tolls.

The first run solution of MO-RBF from Section 5.2 assuming 5% independent failures is evaluated and plotted in Figure 4 for comparison. The same set of S = 350 scenarios are used for evaluation of the toll set obtained from the first run in Section 5.2.

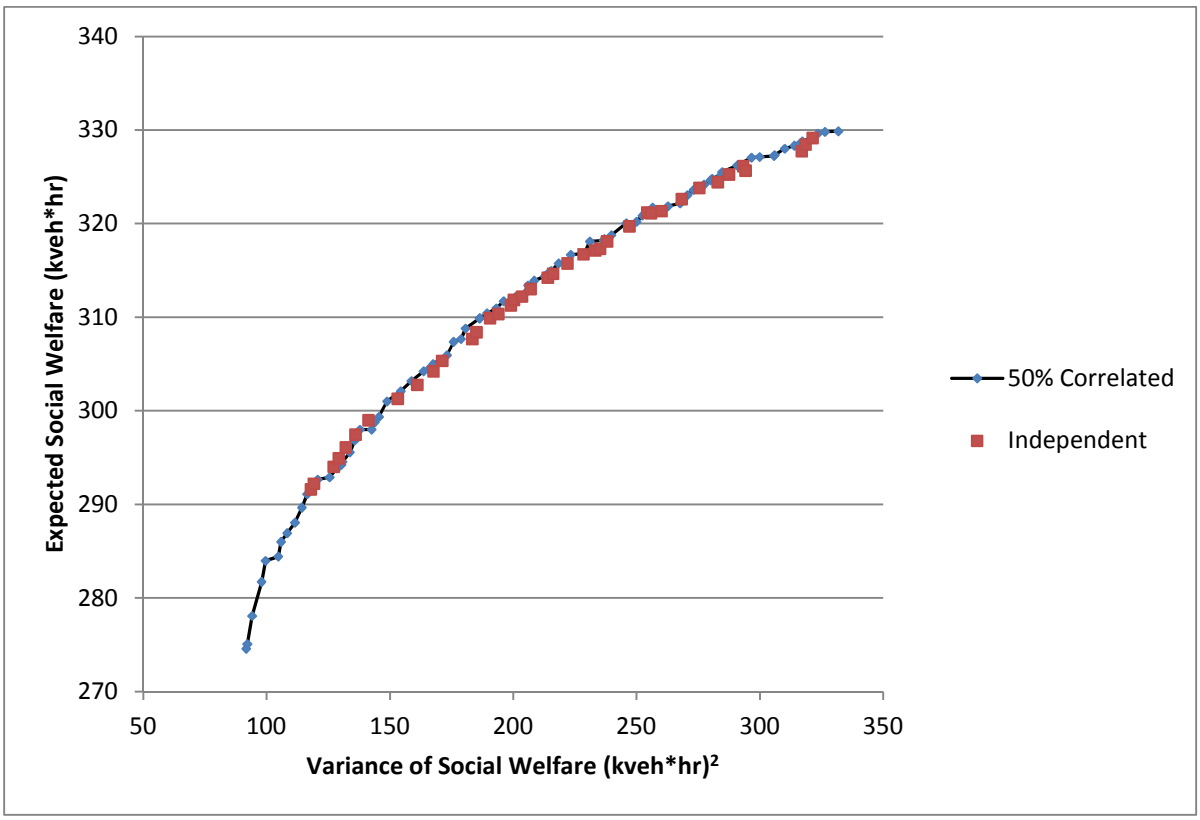

Figure 4. Comparison of MO-RBF Solution with Independent Solution Set under 50% Correlated Link Failure Setting.

As shown in Figure 4, we can observe that the problem assuming independence does not cover the same range of solutions as the MO-RBF solution of the correlated scenarios. The ϵ-indicator for the two solution sets is computed and shown below.

$$I_\epsilon(\Omega_{Independent}, \Omega_{Correlated}) = 1.2858$$

The independent set needs to be improved by 28.6 percent to ϵ-dominate the correlated Pareto set. By assuming independence between link failures, the cost in this example is the loss in ϵ-dominance of 28.6 percent. However, the initial perception from glancing at Figure 4 might appear counter-intuitive to this conclusion. One way of interpreting this conclusion is that the 50% correlated Pareto set is capable of achieving variance of ~90 $(\text{kveh*hr})^2$ and expected social welfare of ~330 kveh*hr. On the other hand, the independent solution can only reach a minimum variance of ~120 $(\text{kveh*hr})^2$. Seen from that point of view, it is quite a significant improvement.

5.5. Quantifying Flexibility to Adapt to Regimes

Chow (2010) argues that transportation agencies can gain significant value by breaking down the static set of network degradation risks into multiple regimes, e.g., different threat levels in homeland security or fire and hurricane seasons. By doing so and solving the robust toll pricing problem for each regime separately, transportation agencies can gain significant value in flexibility. Similarly, Boyles *et al.* (2010) show that day-to-day responsive tolls can be priced more effectively against different regimes than non-responsive tolls. In this test, the value of the flexibility for tolls to respond to changes in the link capacity failure probability regime is quantified in terms of the ϵ-dominance of the Pareto sets.

A high probability failure regime is simulated as a set of scenarios where a link fails with 50% probability, assuming it is independent to any other link. One run of the MO-RBF is conducted with S = 350 scenarios and plotted in Figure 5. The expected social welfare decreases due to the higher likelihood of link failures, and the variance also increases.

The first run solution of MO-RBF from Section 5.2 assuming 5% independent failures is evaluated with the high probability regime scenarios and plotted as points in Figure 5 for comparison. The same set of S = 350 scenarios are used for evaluation of the toll set obtained from the first run in Section 5.2.

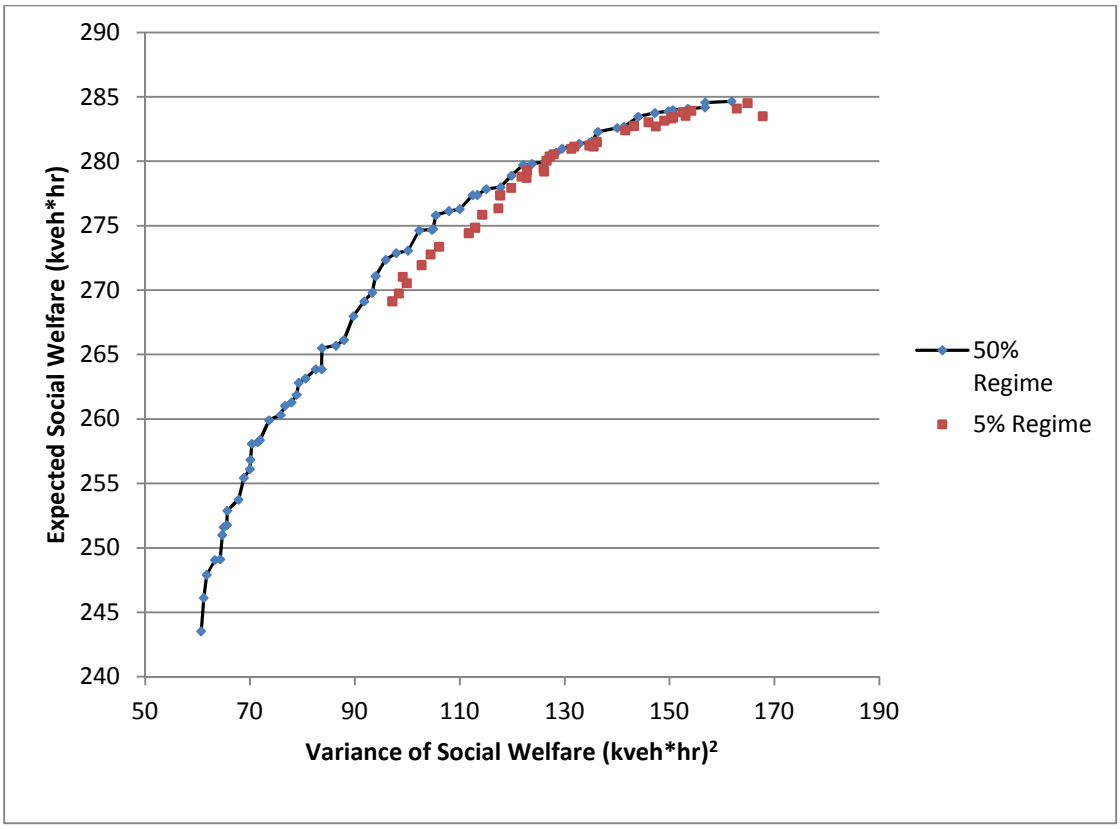

Figure 5. Comparison of Pareto Sets under 50% Failure Regime.

Adjusting the Pareto set to optimally handle a different failure regime results in a more dominant, wider spreading set of solutions. The ϵ-indicator for the two solution sets is computed and shown below.

$$I_\epsilon(\Omega_{5\%}, \Omega_{50\%}) = 1.6009$$

By having the flexibility to adjust the robust toll pricing solution set, the additional value gained is 60.0 percent improvement in ϵ-dominance.

6. CONCLUSION

The goal of this research is to address two problems related to larger robust network design problems to more effectively evaluate operational strategies such as using toll pricing to reduce sensitivity to link disruptions and quantifying the flexibility of adjusting those prices over multiple regimes. A new global heuristic for multiobjective problems is developed based on using radial basis functions as a surrogate model to cheaply search a stochastic local neighborhood for candidates to evaluate. The algorithm is proven to stochastically converge and is compared to existing multiobjective algorithms for four test functions from the literature.

Stochastic link capacity scenarios in a static network assignment are simulated by using a novel scenario simulation model based on converting multivariate Bernoulli random variables to equivalent multivariate normal distributions. Scenarios are then generated using the orthogonal transform approach. This simulation approach captures the simplicity of using Bernoulli variables for failure probabilities while accounting for multiple correlations between different links for evaluating degradations within a network.

Four numerical tests are conducted on the Sioux Falls network. The first test shows that at 350 scenarios and running MO-RBF up to 300 iterations can result in Pareto sets that vary in ϵ-dominance with a standard error less than 2 percent for the Sioux Falls network. The second test compares the MO-RBF Pareto sets with those obtained from a commercial Matlab MOGA function, which indicates that on average the MOGA Pareto sets need to be improved by 24.7 percent in order to ϵ-dominate the MO-RBF Pareto sets for the same set of scenarios. The third test builds on the conclusion drawn from Du and Nicholson (1997) about the error of assuming independent link failures to a real network. In this test, setting a robust toll pricing strategy that assumes link failures are independent even though they are 50% correlated with each other can result in a cost in ϵ-dominance of 28.6 percent. Lastly, the risk management strategy of adapting toll pricing Pareto sets to new information as suggested in Boyles *et al.* (2010) is quantified in terms of the ϵ-indicator. The flexibility to adapt a 5% failure regime solution set to a 50% failure regime solution set results in an improvement in ϵ-dominance of 60.1 percent.

Future work in this direction should consider combining the MO-RBF with evolutionary approaches so that problems with discrete decision variables can also be solved. The approach can be used to solve robust network design problems for larger networks. Many other types of multiobjective problems can be solved using this heuristic. Examples include network design problems in which social justice (road building near major port complexes, for example) or sustainability must be considered. The simulation method presented in this study can be applied to a real example to demonstrate its practicality in modeling how agencies can handle network disaster planning. In terms of modeling multiple regimes, Markov processes can also be considered for the link capacities to derive the value of flexibility in changing toll prices for multi-stage settings. In addition, larger, more typical networks can also be solved using our method though we suspect that the code would have to be re-written on a C or C++ platform and fully optimized, rather than a prototype system in Matlab like the one we developed for this research.

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