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Canonical Representation Of Option Prices and Greeks with Implications for Market Timing

Godfrey Cadogan *

Working Paper

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Abstract

We introduce a canonical representation of call options, and propose a solution to two open problems in option pricing theory. The first problem was posed by (Kassouf, 1969, pg. 694) seeking "theoretical substantiation" for his robust option pricing power law which eschewed assumptions about risk attitudes, rejected risk neutrality, and made no assumptions about stock price distribution. The second problem was posed by (Scott, 1987, pp. 423-424) who could not find a unique solution to the call option price in his option pricing model with stochastic volatility–without appealing to an equilibrium asset pricing model by Hull and White (1987), and concluded: "[w]e cannot determine the price of a call option without knowing the price of another call on the same stock". First, we show that under certain conditions derivative assets are superstructures of the underlying. Hence *any* option pricing or derivative pricing model in a given number field, based on an anticipating variable in an extended field, with coefficients in a subfield containing the underlying, is admissible for market timing. For the anticipating variable is an algebraic number that generates the subfield in which it is the root of an equation. Accordingly, any *polynomial* which satisfies those criteria is *admissible* for price discovery and or market timing. Therefore, at least for empirical purposes, elaborate models of mathematical physics or otherwise are unnecessary for pricing derivatives because much simpler adaptive polynomials in suitable algebraic numbers are functionally equivalent. Second, we prove, analytically, that Kassouf (1969) power law specification for option pricing is functionally equivalent to Black and Scholes (1973); Merton (1973) in an algebraic number field containing the underlying. In fact, we introduce a canonical polynomial representation theory of call option pricing convex in time to maturity, and algebraic number of the underlying–with coefficients based on observables in a subfield. Thus, paving the way for Wold decomposition of option prices, and subsequently laying a theoretical foundation for a GARCH option pricing model. Third, our canonical representation theory has an inherent regenerative multifactor decomposition of call option price that (1) induces a duality theorem for call option prices, and (2) permits estimation of risk factor exposure for Greeks by standard [polynomial] regression procedures. Thereby providing a theoretical (a) basis for option pricing of Greeks, and (b) solving Scott's dual call option problem *a fortiori* with our duality theory *in tandem* with Riesz representation theory. Fourth, when the Wold decomposition procedure is applied we are able to construct an empirical pricing kernel for call option based on residuals from a model of risk exposure to persistent and transient risk factors.

Keywords: number theory; price discovery; derivatives pricing; asset pricing; canonical representation; Wold decomposition; empirical pricing kernel; option Greeks; dual option pricing

JEL Classification Codes: C02, D81, D84, G11-G13, G17

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1 Introduction

We propose a solution to an open problem posed in (Kassouf, 1969, pg. 694) seeking "theoretical substantiation" for his robust option pricing power law which eschewed assumptions about risk attitudes, rejected risk neutrality, and made no assumptions about stock price distribution[1]. Additionally, we propose a solution to a problem posed in (Scott, 1987, pg. 423) who proposed an option pricing model with stochastic volatility where he appealed to an equilibrium asset pricing model in order to explain away a nonuniqueness result he obtained for a call option[2]. Our proposed solution(s), in the realm of algebraic number theory, are robust to assumptions about preferences, stochastic volatility, probability distributions, arbitrage arguments, or equilibrium asset pricing. According to (Clark, 1971, pg. 66) "[f]ield theory is the theoretical background for the theory of equations," Therefore, to the extent that asset pricing models are predicated on equations in the field of real numbers or otherwise is the extent to which they are amenable to analysis under rubric "field theory"–a branch of modern algebra and subfield of algebraic number theory. In particular, this paper provides theoretical justification for market timing with price discovery by and through derivatives[3]. Evidently, price discovery is based on construction of equations to reflect the superstructure of [derivative] assets on which they are based[4]. Accordingly, we provide mathematical justification for price discovery with a superstructure of assets because, according to Clark, the algebraic structure of a field is such that under certain conditions it supports a superstructure. In particular, we introduce a canonical polynomial representation of a call option as the reduced form of stochastic differential equation approaches popularized in the literature.

[1](Kassouf, 1969, pg. 694) concluded his paper by stating

> No pretense is made that the foregoing model "explains" the warrant-common price relationship–but it is hoped that it is a good description that may eventually lead to theoretical substantiation.

For the purpose of this paper we treat a warrant as an option on the "common".

[2](Scott, 1987, pg. 420) explained the problem thus:

> Arbitrage is not satisfactory for the determination of a unique option pricing function in this random variance model. An alternative view of the problem is that the duplicating portfolio for an option in this model contains the stock, the riskless bond, and another call option. We cannot determine the price of a call option without knowing the price of another call on the same stock, but that is precisely the function we are trying to determine.

[3]This paper does not deal with the intricacies of price discovery. However, empirical support for the use of derivative assets as conduit for price discovery was presented in Easley et al. (1998); Pan and Poteshman (2006); Flemming et al. (1996).

[4]See Grossman and Stiglitz (1980)

Our polynomial approach is distinguished from (Hull and White, 1987, pp. 286-287) who used a Taylor series expansion of a call option priced with a Black-Scholes model–conditioned on stochastic volatility which they integrated out to get the call option price. We make no appeal to arbitrage arguments, equilibrium asset pricing arguments or appeal to probability density [or distribution] functions. Similarly, (Hull, 2006, pp. 297-298) proffered a polynomial expansion of Black and Scholes (1973); Merton (1973) formula but did not establish functional equivalence with Kassouf (1969) or show how it could be used to solve Scott (1987) open problem. Nor did Hull (2006) establish a duality theorem for call option based on his polynomial representation.

For application we extend the canonical polynomial representation analysis to a Wold decomposition of call option prices. There, we show how a GARCH(1,1) model can be used to construct an empirical pricing kernel for call option by a signal extraction procedure for unobservable pricing kernel. To the best of our knowledge that procedure is new. However, (Chernov, 2003, pp. 332-333) also assumed an unobservable pricing kernel but used a two stage estimation procedure that involves first stage estimation of parameters from a continuous time asset pricing model. At the second stage, he used an equivalent martingale measure, that includes parameters from the first stage asset pricing model, together with the asset(s) payoff to construct the pricing kernel. He then used a derivative pricing relation that includes parameters of the underlying asset pricing model ion order to derive "independent" equations. Whereupon, "simultaneous equations" are solved to infer the pricing kernel in second stage estimation. Our "two stage" procedure is distinguished because we calibrate second stage *residuals* from a discrete risk pricing model for the underlying asset under consideration, after a first stage Wold decomposition of a call option on the asset.

The rest of this paper proceeds as follows. In section 2 we introduce algebraic equations that support Kassouf's power laws for option prices, and establish their functional equivalence with Black-Scholes-Merton formula. Additionally, we introduce a duality theorem for call option pricing and show how it can be extended to the familiar Snell envelope representation. The main results of the paper are Theorems 2.6, 2.9 and 2.10. Motivated by the power law representation theory, for application we introduce a Wold decomposition for option prices in section 3, as trend or difference stationary (as the case may be) around a convex time trend. Section 4 presents an endogenous pricing kernel for option based on diagnostics from the Wold decomposition. The main result there is Theorem 4.5.

Finally, we conclude with perspectives in section 5.

2 A polynomial representation theory of call option

2.1 Prerequisites

The definitions and theorems that follow were excerpted from myriad sources, and are presented here according as they pertain to terminology used in the sequel.

Definition 2.1 (Field)**.** (Clark, 1971, pg. 66). A field is an algebraic structure in which the four *rational* operations[:] addition, subtraction, multiplication, and division, can be performed and in which these operations satisfy most of the familiar rules of operations with numbers.

Definition 2.2 (Extension field)**.** A field E is called an extension of a field F if F is a subfield of E.

Definition 2.3 (Tower of fields)**.** A sequence of extension fields $F_0 \subset F_1 \subset \ldots \subset F_n$ is called a tower of fields, and F_0 is called the ground field.

Remark 2.1. The nomenclature reflects the fact that F_0 "supports" a superstructure, i.e., a tower, as it were, of fields. (Jacobson, 1951, pg. 103) described this as the *prime field* obtained from intersection of all fields.

Definition 2.4 (Polynomial)**.** A polynomial over a field F is an expression of the form $f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$ where $c_0, c_1, c_2, \ldots, c_n$ are elements of F called coefficients of thee polynomial.

Definition 2.5 (Collection of all polynomials)**.** $F[x]$ is the collection of all polynomials.

Definition 2.6 (Algebraic)**.** Let E be an extension field of the field F. An element α of E is algebraic over F if α is a root of some polynomial with coefficients in F. (Clark, 1971, pg. 88). Alternatively, Let $F \subset E$ be an extension field. An element $\alpha \in E$ is algebraic over F when $f(\alpha) = 0$ for some nonzero polynomial $f(X) \in F[X]$. Otherwise α is transcendental. (Grillet, 2007, pg. 162).

Remark 2.2. Transcendental numbers like e and π are not algebraic, i.e. in and of themselves they are not roots of a finite equation. However, even if π is not algebraic the expression $\cos(k\pi)$ is algebraic for rational values of k because $\cos(k\pi)$ is the root of an equation. See e.g., (Jacobson, 1951, pp. 94-95). (Hilbert, 1998, pg. 3) provides elegant elementary exposition of these concepts.

Most important for this paper is the following

Proposition 2.1 (Interpolation of fields)**.** *(Clark, 1971, pg. 89) If E is an extension field over F and $\alpha \in E$ is algebraic over F, then $F(\alpha)$ is a finite extension of F of degree n where n is the degree of the minimal polynomial for α over F. Furthermore, the set $\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}\}$ is a basis for $F(\alpha)$ over F. That is, $F \subset F(\alpha) \subset E$.*

Proof. See Equation A. □

According to that proposition, the interpolated field $F(\alpha)$ is generated by polynomial powers of an algebraic number. In other words, Kassouf (1969) power law is an admissible option pricing model if it is based on an algebraic number. For instance, assuming deterministic volatility, an option should be priced as a power of standard deviation and the "other" variables would be "coefficients" in the supporting field. This would explain Kassouf (1968) finding of a lag structure for option prices. Additionally, Black and Scholes (1973) and Merton (1973) formula is based on the standard deviation $\sigma \in E$ and *rational products* of transcendental variables with coefficients based in F. So that the latter variables are algebraic transformations of transcendentals[5]. Other useful results from algebraic number theory include:

Definition 2.7 (Monic polynomial)**.** (Pollard and Diamond, 1975, pg. 30) A polynomial is monic if its leading coefficient c_n is 1.

Theorem 2.2 (Unique factorization of polynomials)**.** *Any polynomial $f(x) = c_n x^n + \ldots + c_o$ over F not zero or a constant can be factored into a product $f(x) = c_n \prod_{j=1}^{r} f_j(x)$where the $f_j(x)$ are irreducible monic polynomials over F, determined uniquely except for order.*

Proof. See (Pollard and Diamond, 1975, pg. 30). □

Definition 2.8 (Class of factored polynomials)**.** $\mathfrak{P}$ is the class of all polynomials $p(x)$ which can be factored as in 2.2.

Theorem 2.3 (Uniqueness of minimal polynomial)**.** *If σ is algebraic over F it has a unique minimal polynomial.*

Proof. See (Pollard and Diamond, 1975, pg. 44). □

[5] Haug and Taleb (2008) provide a review of the history of option pricing formulae, and make the case that the Black-Scholes-Merton option pricing formula was "known" to traders long before those authors papers were published.

2.2 Option pricing function space

Take any asset $S \in F$, let σ be a constant standard deviation algebraic in $E \supset F$. According to Proposition 2.1 there exist a finite extension $F(\sigma) \supset F$ with basis $\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$ so that if $\alpha \in F(\sigma)$, there exist coefficients $c_0, c_1, \ldots, c_n$ in F such that for

$$f(\sigma) = c_o + c_1\sigma + c_2\sigma^2 + \ldots + c_n\sigma^{n-1} \tag{2.1}$$

α is a root of the equation. In other words

$$c_o + c_1\alpha + c_2\alpha^2 + \ldots + c_n\alpha^{n-1} = 0 \tag{2.2}$$

The object of price discovery is to find those coefficients $c_i, \quad i = 0, 1, \ldots, n-1$ for which the equation has real roots. In practice we are interested in solutions to the equation

$$(c_o - f(\alpha)) + c_1\sigma + c_2\sigma^2 + \ldots + c_n\sigma^{n-1} = 0 \tag{2.3}$$

where $f(\alpha)$ is a *given value*. In particular $f(\alpha)$ may be a value derived from a no-arbitrage relationship which portends a market equilibrium, and we need to find values of σ that satisfy the equation. If c_i's are known, for arbitrary $\tilde{\alpha} \in E$, then for any factorization $f(\tilde{\alpha}) = g(\alpha)q(\alpha) + r(\tilde{\alpha})$ in which $r(\tilde{\alpha}) \neq 0$ there will be arbitrage opportunities. In particular, there is $\tilde{\alpha} \in E$ for which there is no polynomial $f \in F(\alpha)$ for which it is a root. However, the quotient relationship $g(\alpha)q(\alpha)$ suggests that $f(\tilde{\alpha}) = f(\alpha) + r(\tilde{\alpha})$ where $f(\tilde{\alpha}) = g(\alpha)q(\alpha)$ and $r(\tilde{\alpha})$ is an error term. For instance, the relation holds if $q(\alpha)$ is a *minimal polynomial*. See (Pollard and Diamond, 1975, pg. 44).

2.2.1 Black-Scholes-Merton model

The ubiquitous Black-Scholes-Merton option pricing formula[6] for an European style option–which can only be exercised on terminal date–is typically written as follows

$$C(\sigma|S, K, T, r, t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{2.4}$$

[6]See e.g., (Huang and Litzenberger, 1988, pg. 166) for derivation of formula using lognormal, and preference based assumptions; and (Hull, 2006, Ch. 13) for a taxonomy or no-arbitrage assumptions and applications in different settings.

where

$$t = valuation\ date \tag{2.5}$$
$$S(t) = current\ price\ of\ the\ stock\ on\ valuation\ date \tag{2.6}$$
$$K = strike\ price\ in\ option\ contract \tag{2.7}$$
$$r = risk\ free\ discount\ rate \tag{2.8}$$
$$T = terminal\ date\ of\ contract \tag{2.9}$$
$$\sigma = constant\ standard\ deviation\ of\ stock\ price \tag{2.10}$$
$$\Phi = cumulative\ standard\ normal\ distribution \tag{2.11}$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\Big(\log S(t) - \log K + r(T-t)\Big) \tag{2.12}$$
$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\Big(\log S(t) - \log K - r(T-t)\Big) \tag{2.13}$$

The only "unknown" variable in Equation 2.4 is the "volatility" σ which is forward looking. In the context of field theory this implies that σ is an algebraic number, and the coefficients of Black-Scholes-Merton formula are in a subfield. That is, $\{S,t,r,T\} \in F$, $\sigma \in E$, and BSM formula is a polynomial over F with root(s) in E. The transcendental functions e and Φ are each products of rational numbers S and K so the product are admissible coefficients, and d_1 and d_2 are algebraic, i.e. roots in E for a polynomial over F, to the extent that they depend on $T \in E$. According to Proposition 2.1 the polynomial $C(\cdot)$ over F is algebraic in the extension field E. Since T is known at the time the contract is executed, for all intents and purposes the "algebraic number" generating the polynomial is σ[7]. For instance, if we normalize Equation 2.4 by dividing by K and use the transformation

$$\check{C}(\sigma) = \frac{C(\sigma|\cdot)}{K} - 1 \tag{2.14}$$

Then we must solve

$$\check{C}(\sigma|\cdot) = 0 \tag{2.15}$$

and the σ "roots" can be found by solving a nonlinear equation in a polynomial with coefficients in F. The root(s) portend the "implied volatility" used to gauge

[7] See e.g., (Hilbert, 1998, pg. 3) for generating polynomials.

investor sentiment[8].

2.2.2 Polynomial expansion of Black-Scholes-Merton formula

We start with (Heston, 1993, pg. 330) representation of a generic call option for $\tau = T - t$, and (Apostol, 1967, Thm 6.3, pg. 239)[9]

$$C(\sigma, \tau, S, r, K) = SP_1 - Ke^{-r\tau}P_2 \tag{2.16}$$

where P_1, $P_2 \in \mathfrak{P}$ are probability measures. According to Theorem 2.2 we can write

$$P_i(x) = c_n \prod_{j=1}^{q} f_j(x) \prod_{j=q+1}^{r} f_j(x) \tag{2.17}$$

Assume that

$$f_j(x) = \begin{cases} (1 - \frac{x^2}{2r}) & r > 0 \quad j > q \\ (1 + a_j x) & a_j \in \mathbb{R}, \quad j \leq q \end{cases} \tag{2.18}$$

and that

$$\lim_{n \to \infty} c_n = \frac{1}{\sqrt{2\pi}} \tag{2.19}$$

[8]See (Whaley, 2000, pg. 13) ("[I]mplied volatility is the market's "best" assessment of the *expected* volatility of the underlying stock index over the remaining life of the option") for history of this measure.

[9]Thm 6.3 in Apostol is functionally equivalent to

$$\begin{aligned} N(x) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \\ &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} V(u) \exp(Q(u)) du \end{aligned}$$

for some polynomial $V(u)$ and $Q(u)$. Additionally, (Ambramowitz and Stegun, 1972, pp. 932-933) provide a taxonomy of polynomial representations for $N(x)$.

Hence

$$\lim_{n,r\to\infty} P_i(x) = P_i(x) = \lim_{n,r\to\infty} c_n \prod_{j=1}^{q} f_j(x) \prod_{j=q+1}^{r} f_j(x) \tag{2.20}$$

$$= \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{q} (1 + a_j x) e^{-\frac{x^2}{2}} \tag{2.21}$$

This polynomial expression is functionally equivalent to(Apostol, 1967, Thm. 6.3), and (Hull, 2006, pp. 297-298) who proffered a *polynomial* approximation to Black-Scholes-Merton option pricing formula for an European call option with stock price S, strike price K, and risk free rate r as follows.

$$C(\cdot) = SN(d_1) - Ke^{-r\tau} N(d_2) \tag{2.22}$$

$$N(x) = \begin{cases} 1 - N'(x)\{a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5\} \\ 1 - N(-x) \end{cases} \tag{2.23}$$

where

$$k = \frac{1}{1 + \gamma x} \tag{2.24}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.25}$$

γ is a constant, and d_1 and d_2 are determined as in Equation 2.12 and Equation 2.13. Specifically, in our model

$$P_i(x) = N(x) \tag{2.26}$$

So by plugging in our P_1 and P_2 in (Heston, 1993, pg. 330) we get a polynomial representation for Black-Scholes-Merton formula. This leads to the following

Lemma 2.4 (Polynomial expansion of Black-Scholes-Merton formula)**.** *Let $\mathfrak{P}$ be the class of factored polynomials and $P_j \in \mathfrak{P}$, $i = 1, 2$ where*

$$P_i(x) = c_n \prod_{j=1}^{q} f_j(x) \prod_{j=q+1}^{r} f_j(x)$$

$$f_j(x) = \begin{cases} (1 - \frac{x^2}{2r}) & r > 0 \quad j > q \\ (1 + a_j x) & a_j \in \mathbb{R}, \quad j \leq q \end{cases}$$

Assume that $\lim_{n\to\infty} c_n = \frac{1}{\sqrt{2\pi}}$. *Let* $C(\cdot) = SP_1 - Ke^{-r(T-t)}P_2$ *be the price of a call option* $C(\cdot)$ *and* $N(x)$ *be the cumulative normal distribution evaluated at x. Let*

$$
\begin{aligned}
N(x) &\approx 1 - P_i(x) \\
x &= \frac{\ln \frac{S}{K}}{\sigma\sqrt{T-t}} \\
w &= \frac{r(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Then the price of a Black-Scholes-Merton call option is given by

$$
\begin{aligned}
C(\sigma; S, r, T, t) \approx S(1 - c_n &\prod_{j=1}^{q} f_j(x+w) \prod_{j=q+1}^{r} f_j(x+w) \\
&- Ke^{-r(T-t)} \prod_{j=1}^{q} f_j(x-w) \prod_{j=q+1}^{r} f_j(x-w) \\
&= \psi_0 + \psi_1\sigma + \psi_2\sigma^2 + \ldots + \psi_n\sigma^n
\end{aligned}
$$

where $\psi_k = \psi_k(S, r, T, K, t)$, $k = 1, 2, \ldots, n$

2.2.3 Kassouf's power law model

Kassouf (1969) introduced the following option pricing power law. Let Y be the price of an option, X be the underlying stock price, and K be the strike of the option. Further, let $y = \frac{Y}{K}$ and $x = \frac{X}{K}$ so that

$$
y(z) = (x^z + 1)^{\frac{1}{z}} - 1 \tag{2.27}
$$

is a power law for the underlying option price. The objective is to find a polynomial $y(z)$ with coefficients in F and root in E. That is, the root of $y(z)$ is in E. Kassouf (1969) posited the following

$$\tau = time\ until\ expiration \tag{2.28}$$

$$R = dividend\ yield \tag{2.29}$$

$$DRatio = \frac{\#\ outstanding\ option}{\#\ outstanding\ stocks} \tag{2.30}$$

$$b = slope\ of\ OLS\ model\ fitted\ to\ monthly\ mean\ price \tag{2.31}$$

$$\sigma_b = standard\ deviation\ of\ b \tag{2.32}$$

$$z = \beta_0 + \beta_1 \frac{1}{\tau} + \beta_2 R + \beta_3 DRatio + \beta_4 b + \beta_5 \sigma_b + \beta_6 x + \beta_7 K + \varepsilon \tag{2.33}$$

In that setup if the expiry date is T, then $\tau = T - t$ and z is a "root" of y. We want to find $z \in E$ such that there exist coefficients $c_0, c_1, \ldots, c_n$ in F for which Kassouf's power law in Equation 2.27 holds. We begin with an expansion of $y(z)$ by exploiting the fact that *geometric mean* $\leq$ *arithmetic mean*

$$1.(x^z+1)^{\frac{1}{z}} \leq \frac{1}{\frac{1}{z}}(x^z+1) \tag{2.34}$$

$$= z(x^z+1) \leq z(\frac{x}{z}+1) = (x+z) \tag{2.35}$$

Hence for some differentiable function h we have

$$(x^z+1)^{\frac{1}{z}} = (x+z) - h(z) \tag{2.36}$$

So that a Taylor expansion of h around $z = 0$ yields

$$y(z) = (x^z+1)^{\frac{1}{z}} - 1 \tag{2.37}$$

$$= (x+z) - h(z) - 1 \tag{2.38}$$

$$= (x+z) - [h(0) + \frac{d}{dz}h(z)|_{z=0}\, z + \frac{d^2}{dz^2}h(z)|_{z=0}\, z^2 + \tag{2.39}$$

$$\ldots + \frac{d^n}{dz^n}h(z)|_{z=0}\, z^n] - 1 \tag{2.40}$$

is a polynomial in x and z. In particular, in Equation 2.33 $z = z(\sigma_b, T-t, K, x, r)$. Recall that (Kassouf, 1969, pg. 87) parametrized $z = \alpha + \frac{\beta}{t}$, so assuming arguendo that z is separable in σ_b and the other variables we can rewrite Equation 2.40 as

$$y(z) = a_0 + a_1\tau^{-\lambda} + a_2\sigma_b^1 + \ldots + a_n\sigma_b^{n-1} \tag{2.41}$$

where λ is a constant. (Kassouf, 1969, pg. 691) highlighted the predictive ability of a *forward looking* σ_b on option prices in his model when he opined "[i]f past volatility is a guide to future volatility, this seems reasonable behavior". In which case, $\sigma_b \in E$. The other variables in Kassouf's model reside in F[10]. Thus, Kassouf's model satisfies Proposition 2.1. We summarize the foregoing in a

Lemma 2.5 (Kassouf power law expansion)**.** *Let x be a stock price, $z = z(\sigma_b, T-t, K, x, r)$ and $y(z)$ be a call option on the stock, priced by the equation $y(z) = (x^z+1)^{\frac{1}{z}} - 1$. Let z be separable in σ_b and the other variables indicated. Then*

$$y(z) = a_0 + a_1\tau^{-\lambda} + a_2\sigma_b^1 + \ldots + a_n\sigma^{n-1}$$

2.3 Anatomy of option Greeks

(Kassouf, 1969, pg. 694) concluded his paper by indicating

A. Assumptions about risk attitudes have been purposely avoided.

B. Risk neutrality was rejected by his model.

C. No assumptions were made about stock price distributions.

In fact, he plainly stated

> No pretense is made that the foregoing model "explains" the warrant-common price relationship–but it is hoped that it is a good description that may eventually lead to theoretical substantiation.

Our field theory approach provides a solution to Kassouf's erstwhile open problem by suggesting that z could be fitted as a "naive" polynomial in σ_b and τ as follows

$$z = a_0 + a_1\frac{1}{\tau} + a_2\sigma_b + a_3\sigma_b^2 + \ldots + a_n\sigma_b^{n-1} \tag{2.42}$$

[10] Arguably, dividend yield R is forward looking. However, we assume that agents incorporate that in their assessment of forward looking σ_b through a (Gordon, 1959, pg. 104) type fundamental valuation of stock price or Fama and Babiak (1968) stable dividend result. So the "only" uncertainty in the model is volatility. In any event, an option pricing formula can be derived by assuming no dividends but the same is not true for volatility. See (Hull, 2006, Ch. 13).

Suppose $F(z) \subset E, \quad F \subset F(z) \subset E$ and $\tilde{z} \in E$. Assuming that $\tilde{z}$ is not algebraic in E, we can write

$$y(\tilde{z}) = g(z)q(z) + r(\tilde{z}) \tag{2.43}$$

In the context of Kassouf's model $\tilde{z}$ would be a noisy signal for z, and $r(\tilde{z})$ would be the error term ε. This is an application of the Grossman and Stiglitz (1980) result for partially revealing information in price discovery in a seemingly efficient market. By the same token, in Equation 2.15 we can write the Black-Scholes-Merton formula as a power law

$$\check{C}(\sigma|\cdot) = c_0 + c_1 \frac{1}{\sqrt{T-t}} + c_2\sigma + \ldots + c_n\sigma^{n-1} \tag{2.44}$$

where the c_i's are coefficients in F possibly comprised of a linear combination of S, r, K, $T-t$. In the context of Kassouf (1969); Black and Scholes (1973); Merton (1973) we have, for $\tau = T-t$

$$c_i = c_i(S, r, \tau, K, R) \tag{2.45}$$

In that way, the solution for σ is time dependent, and it depends on the price of the call option C and the c_i's. So we have the *implied volatility*

$$\sigma = \sigma(C, c_i(S, r, \tau, K, R)), \;\; i = 1, 2, \ldots, n-1 \tag{2.46}$$

A cursory inspection of Equation 2.27 and Equation 2.44 shows that each power law is *admissible*, and they are functionally equivalent. Furthermore, each option price fluctuates around a convex trend $\frac{1}{\sqrt{T-t}}$ Therefore, we have just proven the following

Theorem 2.6 (Functional equivalence of Kassouf (1969), Black and Scholes (1973); Merton (1973))**.** *Let σ be the implied volatility of a stock price, C be the price of an option on the underlying, and T be the terminal date of a European call option. Then the Black-Scholes-Merton and Kassouf option pricing formula are each functionally equivalent to a polynomial in implied volatility*

$$C(\sigma|\cdot) = c_0 + c_1\tau^{-\lambda} + c_2\sigma + \ldots + c_n\sigma^{n-1} \tag{2.47}$$

where $\tau = (T-t)$, the coefficients $c_i, \quad i = 1, 2, \ldots, n$ are observables at time t, and λ is a shape parameter.

Proof. Equate coefficients in Lemma 2.4 and Lemma 2.5. □

Remark 2.3. To the extent that σ is a measure of risk, it is evident that the call option price is a nonlinear function of risk, and c_i is a measure of price exposure to the given risk. For instance, c_2 is classic risk exposure, while c_1 could be interpreted as "shape risk" exposure.

This functional equivalence result between Kassouf (1969) and Black and Scholes (1973); Merton (1973) has been bourn out empirically. See e.g., French (1983). Recall that the c_i's include variables in the analysts information set $F \subset E$. So that $\frac{\partial c_i}{\partial S}$ exists.

2.3.1 Identifying Greek risk factors exposures

To obtain estimates for option Greeks[11] we have for *vega* ($\mathscr{V}$), *theta*, *delta*, *gamma* and *rho*

$$\mathscr{V} = \frac{\partial C}{\partial \tau} \tag{2.48}$$

$$= \frac{\partial c_0}{\partial \tau} - \lambda \tau^{-\lambda-1} \frac{\partial c_1}{\partial \tau} + \frac{\partial c_2}{\partial \tau}\sigma + \ldots + \frac{\partial c_n}{\partial \tau}\sigma^{n-1} \tag{2.49}$$

$$\theta = \frac{\partial C}{\partial \sigma} \tag{2.50}$$

$$= c_2 + 2c_3\sigma + 3c_4\sigma^2 + \ldots + (n-1)c_n\sigma^{n-2} \tag{2.51}$$

For *delta* we derive the polynomial

$$\Delta = \frac{\partial C}{\partial S} \tag{2.52}$$

$$= \frac{\partial c_0}{\partial S} + \frac{\partial c_1}{\partial S}\tau^{-\lambda} + \frac{\partial c_2}{\partial S}\sigma + \ldots + \frac{\partial c_n}{\partial S}\sigma^{n-1} \tag{2.53}$$

[11] For a thorough review of this concept see (Hull, 2006, Ch. 15), and Passarelli (2008).

Similarly for option *gamma* we have

$$\gamma = \frac{\partial^2 C}{\partial S^2} \tag{2.54}$$

$$= \frac{\partial^2 c_0}{\partial S^2} + \frac{\partial^2 c_1}{\partial S^2}\tau^{-\lambda} + \frac{\partial^2 c_2}{\partial S^2}\sigma + \ldots + \frac{\partial^2 c_n}{\partial S^2}\sigma^{n-1} \tag{2.55}$$

$$\rho = \frac{\partial C}{\partial r} \tag{2.56}$$

$$= \frac{\partial c_0}{\partial r} + \frac{\partial c_1}{\partial r}\tau^{-\lambda} + \frac{\partial c_2}{\partial r}\sigma + \ldots + \frac{\partial c_n}{\partial r}\sigma^{n-1} \tag{2.57}$$

In practice, σ is unobservable and changes with time. Therefore, it can be estimated with the class of ARCH-type models introduced by Engle (1982) and Bollerslev (1986)[12].

Perhaps most important is the inherent decomposition of option Greeks with a multifactor representation. In particular, under our approach regression results provide estimates of the following *factor exposures*, i.e.

$$c_i^{\Delta} = \frac{\partial c_i}{\partial S} \tag{2.58}$$

$$c_i^{\gamma} = \frac{\partial^2 c_i}{\partial S^2} \tag{2.59}$$

$$c_i^{\tau} = \frac{\partial c_i}{\partial \tau} \tag{2.60}$$

$$c_i^{\sigma} = \frac{\partial c_i}{\partial \sigma} \tag{2.61}$$

$$c_i^{\rho} = \frac{\partial c_i}{\partial r} \tag{2.62}$$

Thus we have the following

Theorem 2.7 (option Greeks Decomposition)**.** *Let $C(\sigma|\, S,r,T,t)$ be the price of a call option, and σ be the volatility of the underlying. Let $c_i(S,r,T,t)$ be the i-th risk exposure factor for the option. Then we have the following factor decomposition*

[12] See Engle (2001, 2004) for review of these models.

for a call option and its associated Greeks

$$C(\sigma|\cdot) = c_0 + c_1\tau^{-\lambda} + c_2\sigma + \ldots + c_n\sigma^{n-1} \tag{2.63}$$

$$\mathscr{V} = \frac{\partial C}{\partial \tau} = \frac{\partial c_0}{\partial \tau} - \lambda\tau^{-\lambda-1}\frac{\partial c_1}{\partial \tau} + \frac{\partial c_2}{\partial \tau}\sigma + \ldots + \frac{\partial c_n}{\partial \tau} \tag{2.64}$$

$$\theta = \frac{\partial C}{\partial \sigma} = c_2 + 2c_3\sigma + 3c_4\sigma^2 + \ldots + (n-1)c_n\sigma^{n-2} \tag{2.65}$$

$$\Delta = \frac{\partial C}{\partial S} = \frac{\partial c_0}{\partial S} + \frac{\partial c_1}{\partial S}\tau^{-\lambda} + \frac{\partial c_2}{\partial S}\sigma + \ldots + \frac{\partial c_n}{\partial S}\sigma^{n-1} \tag{2.66}$$

$$\gamma = \frac{\partial^2 C}{\partial^2 S} = \frac{\partial^2 c_0}{\partial S^2} + \frac{\partial^2 c_1}{\partial S^2}\tau^{-\lambda} + \frac{\partial^2 c_2}{\partial S^2}\sigma + \ldots + \frac{\partial^2 c_n}{\partial S^2}\sigma^{n-1} \tag{2.67}$$

$$\rho = \frac{\partial C}{\partial r} = \frac{\partial c_0}{\partial r} + \frac{\partial c_1}{\partial r}\tau^{-\lambda} + \frac{\partial c_2}{\partial r}\sigma + \ldots + \frac{\partial c_n}{\partial r}\sigma^{n-1} \tag{2.68}$$

Remark 2.4. These models can be estimated by a 2SLS polynomial regression on the nonlinear risk factors $\sigma^k,\ k = 1,2,\ldots,n$ as follows. First, run a regression on the τ variable. Second, run a regression of the residuals of the first stage on the nonlinear σ^k's to get least squares estimates of the various risk exposures. Furthermore, to address any potential multicollinearity problems, a principal component analysis would produce orthogonal linear combinations of factors that enhance the multifactor representation, and facilitate statistical inference. In order not to overload the paper we did not present those theories here. However, the interested reader is referred to (McCullagh and Nelder, 1989, pg. 69) and (Weisberg, 2005, Ch. VI) for theoretical ramifications of polynomial regressions, and (Rao, 1973, pg. 590, §8g.2) for a succinct and rigorous presentation on principal components analysis.

2.3.2 A dual option price theory

The polynomial representation for option prices induce shadow option price representation for the Greeks. For instance, Equation 2.53 and Equation 2.55 generate shadow call option[13], and Equation 2.51 generates a call option of its own.

[13] Our call option on the γ-process is distinguished from the "Variance-Gamma" process introduced by Madan and Seneta (1990); Madan and Milne (1991) for subordinate Brownian motion. The latter has to do with jump processes used to price option in a Black-Scholes-Merton setting.

To see that let

$$\tilde{c}_i = \frac{\partial c_i}{\partial S} \tag{2.69}$$

$$\tilde{\tilde{c}}_i = \frac{\partial \tilde{c}_i}{\partial S} \tag{2.70}$$

There exist $\tilde{C}_\Delta(\sigma)$ and $\tilde{\tilde{C}}_\gamma(\sigma)$ such that

$$\frac{\partial C}{\partial S} = \tilde{C}_\Delta(\sigma) = \tilde{c}_0 + \tilde{c}_1 \tau^{-\lambda} + \tilde{c}_2 \sigma + \ldots + \tilde{c}_n \sigma^{n-1} \tag{2.71}$$

$$\frac{\partial^2 C}{\partial S^2} = \tilde{\tilde{C}}_\gamma(\sigma) = \tilde{\tilde{c}}_0 + \tilde{\tilde{c}}_1 \tau^{-\lambda} + \tilde{\tilde{c}}_2 \sigma + \ldots + \tilde{\tilde{c}}_n \sigma^{n-1} \tag{2.72}$$

The general coefficient on the right hand side of Equation 2.51 is $(j-1)c_j, \;\; j = 2, 3, \ldots, n$. Multiplication by σ gives

$$\sigma\theta = c_2\sigma + 2c_3\sigma^2 + \ldots + (n-1)c_n\sigma^{n-1} \tag{2.73}$$

So that if $\breve{c}_j = (j-1)c_j$ we have

$$\frac{\partial C}{\partial \sigma} = \breve{C}(\sigma|\ \theta) = c_0 + c_1\tau^{-\lambda} + \breve{c}_2\sigma + \ldots + \breve{c}_n\sigma^{n-1} \tag{2.74}$$

$$= c_0 + c_1\tau^{-\lambda} + \theta\sigma \tag{2.75}$$

By the same token we have

$$\frac{\partial C}{\partial \tau} = C(\sigma|\ \mathscr{V}) = \frac{\partial c_0}{\partial \tau} + \left(\frac{\partial c_1}{\partial \tau} - \frac{\lambda c_1}{\tau}\right)\tau^{-\lambda} + \frac{\lambda c_2}{\partial \tau}\sigma + \ldots + \frac{\partial c_n}{\partial \tau}\sigma^{n-1} \tag{2.76}$$

$$= \bar{c}_0 + \bar{c}_1\tau^{-\lambda} + \bar{c}_2\sigma + \ldots + \bar{c}_n\sigma^{n-1} \tag{2.77}$$

The canonical polynomial representation for option is regenerative in that option Greek have a call option representation feature[14]. In other words, in our model an analyst could price and trade option on the Greeks[15]. Perhaps most important

[14] Arguably, this is a derivative free result. Cf. Benth et al. (2010).

[15] For instance, (Passarelli, 2008, pg. xvi) states:

> Option traders must consider the time period in question, the volatility expected during the period, interest rate, and dividends. Along with the stock price, these factors makeup the dynamic component of an option's value. These individual factors can be isolated, measured, and exploited. Incremental changes in any of these elements provide opportunity for option traders. *Option greeks* is the term

is the representation in Equation 2.75 which plainly shows that the coefficient for σ is the θ value for *another* call option, which we will call a *conjugate* or *option dual*. In particular, given the reduced form it suggests that the θ value for a call option in Equation 2.47 is c_2. Thus, we have just proven the following

Theorem 2.8 (Well defined call option representation)**.** *Let*

$$C(\sigma|\cdot) = c_0 + c_1\tau^{-\lambda} + c_2\sigma + \ldots + c_n\sigma^{n-1} \tag{2.78}$$

be the canonical polynomial representation of a call option. Then $C(\sigma|\cdot)$ is well defined if

$$2c_3\sigma^2 + \ldots + (n-1)c_n\sigma^{n-1} = 0 \tag{2.79}$$

Whereupon

$$\gamma = -\Sigma_{j=4}^{n}(j-1)c_j\sigma^{j-3} \tag{2.80}$$

Proof. In order for $\frac{\partial C}{\partial \sigma} = c_2$ Equation 2.79 must hold. Whence for $\sigma \neq 0$ Equation 2.80 follows from definition of $\gamma = \frac{\partial^2 C}{\partial^2} = 2c_3$. □

Remark 2.5. Naive partial differentiation of Equation 2.47 with respect to σ gives $c_2 = \frac{\partial C}{\partial \sigma}$ which does not yield the result in Equation 2.79. So that result is a hypothesis to be tested. If anything, the result suggests that the canonical option pricing model depends on σ and possibly σ^2 because $2c_3 = \frac{\partial C}{\partial \sigma^2}$ is the option γ. In any case, (Hull, 2006, pg. 359) plainly states that "when $[\theta]$ is large and *positive*, $[\gamma]$ of a portfolio tends to be large and *negative*". That empirical regularity is clearly reflected in Equation 2.80.

The well defined prerequisites suggest the following

Theorem 2.9 (Call option duality)**.** *Let*

$$C(\sigma|\cdot) = c_0 + c_1\tau^{-\lambda} + c_2\sigma + c_3\sigma^2 \ldots + c_n\sigma^{n-1} \tag{2.81}$$

used for the way the incremental changes in factors affecting an option price are measured. Because of these other influences, direction is not the only tradeable element of a forecast. Time, volatility, interest rates–these can all be traded using option.

be the canonical polynommial representation of a call option. Then there exist a dual call option

$$C^*(\sigma|\ \theta,\gamma) = c_0^* + c_1^*\tau^{-\lambda} + \theta\sigma + \frac{1}{2}\gamma\sigma^2 + c_4^*\sigma^3 + \ldots + c_n^*\sigma^{n-1} \tag{2.82}$$

where θ, and γ are the Greeks for $C(\sigma|\ \cdot)$.

Theorem 2.10 (Call option representation for Greeks)**.** *Let $G = \{\Delta, \theta, \gamma, \mathscr{V}, \rho\}$ be the set of Greeks for a call option C^* with algebraic volatility number $\sigma \in E$. Let $\{c_0, c_1, \ldots, c_n\} \in F$ and $F(\sigma)$ be the subfield generated by polynomials in σ so that $F \subset F(\sigma) \subset E$. Assume that for*

$$g_1,\ g_2 \in G\ c_i^{g_1} \neq c_i^{g_2},\ g_1 \neq g_2 \tag{2.83}$$

$$C(\sigma) = c_0 + c_1\tau^{-\lambda} + c_2\sigma + c_3\sigma^2 + \ldots + c_n\sigma^{n-1} \tag{2.84}$$

Then for any $g \in G$ there exist a call option $C_g(\sigma)$ such that

$$C(g,\sigma) = c_0^g + c_1^g\tau^{-\lambda} + c_2^g\sigma + c_3^g\sigma^2 + \ldots + c_n^g\sigma^{n-1} \tag{2.85}$$

$$c_2^g = \theta = \frac{\partial C^*}{\partial \tau} \quad c_3^g = \frac{1}{2}\gamma = \frac{1}{2}\frac{\partial C^*}{\partial \sigma} \tag{2.86}$$

where the coefficients c_i^g correspond to $g \in G$.

2.4 A function space solution to Scott (1987) call option dual problem

(Scott, 1987, pg. 423) introduced an option pricing model with stochastic volatility which produced a nonunique result he could not solve without appealing to an equilibrium asset pricing argument setforth in Hull and White (1987). However, Theorem 2.9 suggests that there is an operation which when performed on a call option $C(\sigma)$ results in another call option, i.e. a dual, $C^*(\sigma)$. Accordingly, we setup the following topology[16]. Let $\mathscr{C}$ be the space of call option in $F(\sigma)$ so that

$$\mathscr{C} = \{C(\sigma)|\ C(\sigma) = c_0 + c_1\tau^{-\lambda} + c_2\sigma + \Sigma_{j=3}^{n-1} c_j\sigma^{j-1} \in F\} \tag{2.87}$$

[16]See (Brown and Ross, 1988, pg. 5) for operator theory and functional analysis of option pricing.

Let $\mathscr{C}^*$ be the dual space to $\mathscr{C}$, and T be an operator defined on $F(\sigma)$. By definition $\mathscr{C}^*$ is the space of linear operators defined on $\mathscr{C}$. So that

$$T : \mathscr{C} \to \mathscr{C}^* \tag{2.88}$$

and T^* is an operator defined on $\mathscr{C}^*$. Define a norm on $F(\sigma)$ so that

$$\|C(\sigma)\|^2_{F(\sigma)} =< C(\sigma), C(\sigma) > \tag{2.89}$$

Thus the Banach space

$$\mathfrak{H} = (\mathscr{C}, \|\|) \tag{2.90}$$

is a Hilbert space whose norm is an inner product. We state the following

Theorem 2.11 (Riesz representation)**.** *Assume that T^* is a continuous linear operator defined on the dual space$\mathfrak{H}^* = (\mathscr{C}^*, \|\|)$. Then there is a unique $C(\sigma) \in \mathfrak{H}$ such that $T^*(C^*) =< C^*, C >$ for all $C^* \in \mathfrak{H}^*$. Furthermore, $\|C\| = \|T^*\|$.*

Proof. See (Reed and Simon, 1980, pg. 43). □

Thus, our call option duality Theorem 2.9, in tandem with Riesz representation theorem, solves the Scott‘s nonunique call option problem without resort to equilibrium asset pricing models.

2.4.1 Snell envelope representation of dual call option

However, we go further and establish functional equivalence with the duality approach taken by Snell‘s envelope method. Instead of a “tower of fields” we use a filtration $F(\sigma) = \{\mathscr{F}_t;\, t \geq 0\}$ where $\mathscr{F}_s \subseteq \mathscr{F}_t;\, s \leq t$, and $E = \mathscr{F}_\infty$. We begin with the following

Lemma 2.12 (Martingale decomposition of call option on Greeks)**.** *Let $F \subset F(\sigma) \subset E$ and P be a probability measure on E for a finite horizon $[0,T]$. Let Ω be a sample space for states of nature, and $(\Omega, E, \mathscr{F}_t, P)$ be a probability space. Assume that*

$$A_t = c_0 + c_1 \left(\frac{1}{T-t}\right)^{\lambda} \tag{2.91}$$

$$M_t = \frac{\partial C}{\partial t}\sigma + \frac{1}{2}\frac{\partial C}{\partial \sigma}\sigma^2 \tag{2.92}$$

Then there exist an option $\tilde{C}$ such that

$$\tilde{C}_t = A_t + M_t \tag{2.93}$$

is a martingale decomposition.

Proof. By virtue of the regenerative property in Theorems 2.9 and 2.10 $A-t$ and M_t are call option. By the convexity property of call option, and the regenrative property, the sum $\tilde{C}_t = A_t + M_t$ is also a call option. However, by construction A_t is increasing in t, and if $\mathbb{E}^P$ is an expectation operator with respect to P, then by definition of the extension field E,

$$\mathbb{E}^P[M_t \in E|\, F(\sigma)] = M_t(\sigma). \tag{2.94}$$

So that

$$\mathbb{E}^P[\tilde{C}_t \in E|\, F(\sigma)] = \mathbb{E}^P[A_t \in E|\, F(\sigma)] + \mathbb{E}^P[M_t \in E|\, F(\sigma)] \tag{2.95}$$

$$= A_t + M_t(\sigma) \tag{2.96}$$

Because A_t is an increasing process, and $M_t(\sigma)$ is a martingale under P, the relation $C_t(\sigma) = A_t + M_t(\sigma)$ is a Doob-Meyer martingale decomposition. □

The lemma allows us to pose the call option problem in the realm of the more familiar primal-dual relationship using Snell's envelope[17]. We state the following without proof

Proposition 2.13 (Primal-Dual Problem)**.** *Let $\Gamma[0,T-t]$ be a set of stopping times over the interval $[0,T-t]$, and $\mathscr{A}$ be the class of increasing processes. Then the primal-dual problem is defined thus.*

$$\textit{Primal}: \quad C_0 = \sup_{\Gamma[0,T-t]} \mathbb{E}_0^P[C(\sigma) \in E]$$

$$\textit{Dual}: \quad C_t = \inf_{A\in\mathscr{A}}\{A_t + \mathbb{E}_t^P[\max_{u\in[t,T]}[C_u - A_u]|\, F(\sigma)]\}$$

Proof. See Chow and Robbins (Chow and Robbins), and (Wang and Caflish, 2010, pp. 4-5). □

In other words, we have the following

[17] See Snell (1952).

Lemma 2.14. *Let $\tilde{C}^1_u(\theta)$ and $\tilde{C}^2_u(\gamma)$ be the regenerative call option generated by the Greeks θ and γ, respectively. Then for any call option $C_t(\sigma)$ we have the primal-dual representation*

$$C_t(\sigma) = \inf_{A\in\mathscr{A}}[c_0 + c_1(T-t)^{-\lambda} + \mathbb{E}^P[\max_{u\in\Gamma[0,T-t]} C_u(\sigma) - \{c_0 + c_1(T-u)^{-\lambda}\} \tag{2.97}$$

$$= \inf_{A\in\mathscr{A}}[c_0 + c_1(T-t)^{-\lambda} + \mathbb{E}^P[\max_{u\in\Gamma[0,T-t]}[\sigma\tilde{C}^1_u(\theta) + \frac{1}{2}\sigma^2\tilde{C}^2_u(\gamma)] \tag{2.98}$$

Proof. See Proposition 2.13. □

3 Wold decomposition of option prices

So far, out results have been deterministic. However, in practice we deal with samples of option prices. So we need to modify our results accordingly[18]. We start with a sample space for the laws of nature, depicted by Ω. A sample point in Ω is depicted by ω, and an event A is a Borel measureable set comprised of points, i.e., subsets in Ω. The set of possible realizations of a call option $C(\omega)$; and a probability measure P on Ω. The fields F are replaced by time dependent fields $\mathscr{F}_s \subset \mathscr{F}_t,\ \ s < t$, and the extended field E is replaced by $\mathscr{F}_\infty$. Thus, for fixed t the quantity $C_t(\omega)$ is a random variable. Whereas for fixed ω the quantity $\{C_t(\omega); \mathscr{F}_t,\ t \geq 0\}$ is a stochastic process. Furthermore instead of the regular addition and subtraction in definition 2.1 we use set theoretic notation to account for the "fields of information" flowing over Ω. In particular, a [finite] field is characterized by a union [addition] and intersection [multiplication] of sets. Now this concept is extended to the notion of a measure over countable unions and intersections of Borel measurable subsets of Ω[19]. We discretize the finitie horizon $[0,T$ containing time until expiry with a partition of dyadic points $t^{(n)}_k$, $k = 1,2,\ldots,2^n$ so that $\mathscr{F}_{t^{(n)}_k} \subseteq \mathscr{F}_{t^{(n)}_{k+1}}$. In that way for any time $t^{(n)}_k \leq t < t^{(n)}_{k+1}$ we have $\lim_n \to t^{(n)}_k = t$. Thus, the stochastic process $C_t(\omega)$ is right continuous with left hand limit (RCLL). This entails extension of the field concept by replacing the "tower of fields" in definition 2.3 with a filtration $\mathbb{F}$ of σ-field $\mathscr{F}$. In standard texts like (Karatzas and Shreve, 1991, pg. 10) the "ground field" in definition 2.1, and our $\mathscr{F}_0$ here, con-

[18] See (Gikhman and Skorokhod, 1969, Ch. III) for an excellent review of the ensuing concepts.
[19] See (Kolmogorov, 1956, pp. 16-18) for elementary discussions on these concepts.

tains the "P-negligible sets". Together with the RCLL property, these are known as "the usual conditions" for a filtration of fields. In which case, our model is extended to the probability space $(\Omega, \mathscr{F}_t, \mathbb{F}, P)$. We begin by stating, without proof, the well known

Theorem 3.1 (Wold Decomposition Theorem)**.** *Let $\xi(t,\omega)$ be a stationary sequence for $t = 0, \pm 1, \pm 2, \ldots$, and let H_ξ be the closed linear hull in the space of squared integrable Lebesgue functions, $L^2(\Omega, \mathscr{F}, \mathbb{F}, P)$, generated by ξ. Furthermore, let $H_\xi(t)$ be the closed linear hull generated by ξ for $n \leq t$. Let $H_\xi^S(t) = \cap_t H_\xi(t) \subset \mathbb{F}$. Then an arbitrary sequence $\xi(t,\omega) \in L^2(\Omega, \mathscr{F}, \mathbb{F}, P)$ has a unique decomposition of the form*

$$\xi(t,\omega) = \xi_S(t) + \eta(t,\omega) \tag{3.1}$$

where ξ and η are uncorrelated sequences that are subordinate to $\xi(t,\omega)$, $\xi_S(t)$ is deterministic, and $\eta(t,\omega)$ is a MA(∞) process.

Proof. See (Brockwell and Davis, 1987, pg. 180) and (Gikhman and Skorokhod, 1969, pg. 243). □

Under that set up, Equation 2.44 and Equation 2.27 provide a basis for Wold decomposition of call option prices. Specifically, after the [convex] deterministic trend component is removed by a first difference we are left with a moving average of volatility, which, by definition, is an ARCH-type process. Formally, let $\tilde{C}_t(\sigma,\omega)$ be fluctuations of option prices around trend at time t. Then we can rewrite the expression in Theorem 2.6 as

$$\check{C}_t(\sigma,\omega) = \text{trend}_t + \tilde{C}_t(\sigma,\omega) \tag{3.2}$$

Let Δ be a difference operator and L be a lag operator. So that

$$\Delta\check{C}_t(\sigma,\omega) = \check{C}_t(\sigma,\omega) - \check{C}_{t-1}(\sigma,\omega) = \check{C}_t(\sigma,\omega) - L\check{C}_t(\sigma,\omega) \tag{3.3}$$

$$= (1-L)\check{C}_t(\sigma,\omega) = \text{trend}_t - \text{trend}_{t-1} + \Delta\check{C}_t(\sigma,\omega) \tag{3.4}$$

Because $\tilde{C}_t(\sigma,\omega)$ is stationary around trend we have a Wold decomposition

$$\Delta\tilde{C}_t(\sigma,\omega) = \Delta\tilde{C}_t(\sigma)_S + \eta_t(\omega) \tag{3.5}$$

in which η has a $MA(\cdot)$ representation. For example, we write

$$\eta_t(\omega) = \Sigma_{j=1}^{\infty} a_t \psi_t \tag{3.6}$$

where $E[\psi_t] = 0$ and $E[\psi_t^2] < \infty$

$$\tag{3.7}$$

If the trend is linear then $\tilde{C}$ is stationary. If it is quadratic, then the *difference stationary* process $\Delta\tilde{C}_t(\sigma, \omega)$ is analyzed[20]. For more complicated trends, more sophisticated "differencing" or filtering schemes may be required before the MA term can be analyzed. (Kassouf, 1976, pg. 305) presented empirical evidence of a lag structure in option prices which lends credence to our Wold decomposition hypothesis.

4 An endogenous pricing kernel for option

It is axiomatic that fluctuations in detrended option prices are linked intertemporally by a stochastic discount factor or pricing kernel that reflects, *inter alia* the time value of money. Let m_t be a pricing kernel. No arbitrage arguments[21] imply that

$$E[\tilde{C}_{t+1}|\mathscr{F}_t] = E[m_{t+1}\tilde{C}_t|\mathscr{F}_t] \tag{4.1}$$

which can be rewritten as

$$\tilde{C}_{t+1} = m_{t+1}\tilde{C}_t + \vartheta_{t+1} \tag{4.2}$$

for some error term ϑ. Additionally

$$E[m_{t+1}|\mathscr{F}_t] = E[\frac{\tilde{C}_{t+1}}{\tilde{C}_t}|\,\mathscr{F}_t] \tag{4.3}$$

[20] $\Delta\tilde{C}_t$ is differenced to get at its stationary part $\Delta^2\tilde{C}_t$

[21] See (Campbell et al., 1997, pg. 295) for definition and derivation of *pricing kernel*.

However

$$Var(\frac{C_{t+1}}{C_t}|\,\mathscr{F}_t) = E[\{\frac{C_{t+1}-E[C_{t+1}]}{C_t}\}^2|\,\mathscr{F}_t] = \frac{\sigma_t^2}{C_t} \tag{4.4}$$

which can be rewritten as

$$Var\{\tilde{C}_{t+1}|\,\mathscr{F}_t\} = \tilde{C}_t\sigma_t^2 \tag{4.5}$$

By definition, this is functionally equivalent to Engle's (1982) ARCH specification, for fluctuations $\tilde{C}_t$ around a trend, as follows. Let ξ_t be the unobservable innovation in detrended claims, such that $Var\{\xi_t\} = \sigma_t^2$, and write the seperable process

$$\tilde{C}_{t+1} = \sqrt{|\tilde{C}_t|}\xi_t + \vartheta_{t+1} \tag{4.6}$$

So that unconditionally

$$E[\tilde{C}_{t+1}] = E[\sqrt{|\tilde{C}_t|}]E[\xi_t] = 0 \tag{4.7}$$

By hypothesis $E[\tilde{C}_{t+1}] = 0$, so that

$$E[\xi_t] = 0 \tag{4.8}$$

Undeniably, the detrended conditional option price process is stochastic by virtue of being a function of ξ-innovations. That is

$$E[\tilde{C}_{t+1}|\,\mathscr{F}_t] = \sqrt{|\tilde{C}_t|}\xi_t \tag{4.9}$$

Because $\lim_{t\to\infty} m_t = 1$ we can write $m_t = 1 + u_t$ where P-$\lim_t u_t = 0$. Therefore, m_t has a Wold decomposition. See section subsubsection 4.1.1, *infra*. That is, it can be represented as a MA(∞) process. Specifically, since m_t is unobservable, let it be measured with error given by η_t. So we observe

$$\tilde{m}_t = m_t + \eta_t \tag{4.10}$$

and the unconditional next period option price is now

$$\tilde{C}_{t+1} = m_{t+1}\tilde{C}_t + \eta_{t+1}\tilde{C}_t \tag{4.11}$$

This has the same functional form as Equation 4.2 with

$$\tag{4.12}$$

$$\vartheta_{t+1} = \eta_{t+1}\tilde{C}_t \tag{4.13}$$

To see that, since $E[\eta_{t+1}|\,\mathscr{F}_t]$ the conditional variance is

$$Var\{\tilde{C}_{t+1}|\,\mathscr{F}_t\} = E[\{\tilde{C}_{t+1} - E[\tilde{C}_{t+1}|\,\mathscr{F}_t]\}^2] \tag{4.14}$$

$$= \tilde{C}_t Var(\vartheta_{t+1}) = \tilde{C}_t^2 \sigma^2_{\eta_{t+1}} \tag{4.15}$$

Let

$$\varepsilon_{t+1} = \sqrt{|\tilde{C}_t|}\eta_{t+1} \tag{4.16}$$

So that

$$Var(\varepsilon_{t+1}) = |\tilde{C}_t|\sigma^2_{\eta_{t+1}} \tag{4.17}$$

This implies that we can write

$$\tilde{C}_{t+1} = \sqrt{|\tilde{C}_t|}\varepsilon_{t+1} + \vartheta_{t+1} \tag{4.18}$$

$$= \tilde{C}_t\eta_{t+1} + \vartheta_{t+1} \tag{4.19}$$

which has the same form as Equation 4.6. It is precisely at this point that (Engle, 1982, pg. 988) realized that that autoregressive specification could lead to a variance of zero or infinity, and he suggested the autoregressive conditional heteroskedasticity (ARCH) correction

$$\tilde{C}_{t+1} = \eta_{t+1}\sqrt{\sigma^2_{\tilde{C}_t}} \tag{4.20}$$

$$\sigma^2_{\tilde{C}_t} = \theta_0 + \theta_1\tilde{C}^2_{t-1} \tag{4.21}$$

with the proviso that, unconditionally, $E[\eta_t] = 0$ and $Var(\eta_t) = 1$. It should be noted that the foregoing specification handles negative values for incremental op-

tion price through the sign of η_t. Thus, we have just proven the following

Theorem 4.1 (ARCH in Detrended option Prices)**.** . *Let $\tilde{C}_t$ be the stationary part of a Wold decomposition of a call option price at time t. Let $\mathscr{F}_{t-1}$ be the information set available at time $t-1$, and m_t be an unobservable price kernel that links the option prices between times t and $t+1$ such that $\tilde{C}_{t+1} = m_{t+1}\tilde{C}_t$. Let the conditional variance of option prices in period t be*

$$Var(\tilde{C}_t | \mathscr{F}_{t-1}) = \tilde{C}_{t-1}\sigma_t^2$$

Suppose that $\tilde{m}_t = m_t + \eta_t$ is observed, but true m_t and measurement error η_t are unobservable. Let $E[\eta_t] = 0$ and $Var(\eta_t) = 1$. Then trend stationary call option prices follow an ARCH process

$$\tilde{C}_t = \eta_t \sqrt{\sigma^2_{\tilde{C}_{t-1}}}$$
$$\sigma^2_{\tilde{C}_t} = \theta_0 + \theta_1 \tilde{C}^2_{t-1}$$

Remark 4.1. This Theorem was derived by using a fairly standard signal-noise parametrization for the pricing kernel.

At Engle's suggestion, Bollerslev proposed a more parsimonious model to mitigate the long lag structure encountered in ARCH models in practice. See (Bollerslev, 1986, pp. 307, 308). Instead of the ARCH process, Bollerslev introduced a Generalized ARCH process which, in the context of our detrended option price process, implies the following

Corollary 4.2 (GARCH(1,1) Detrended Option Process)**.** . *Let η_t, the measurement error in observed pricing kernel for call option prices at time t, be distributed with mean zero and unconditional variance $Var(\tilde{C}_t) = \sigma^2_{\tilde{C}_t}$. Then a GARCH(1,1) process is admissible for evolution of the dynamics of detrended call option prices. In particular,*

$$\sigma^2_{\tilde{C}_t} = \alpha_1 \varepsilon^2_{t-1} + \beta_1 \sigma^2_{\tilde{C}_{t-1}} \tag{4.22}$$

Where

$$\alpha_1 + \beta_1 < 1 \tag{4.23}$$

Remark 4.2. By definition in Equation 4.16, ε_t is a convex function of $\tilde{C}_t$. Furthermore, the quantity $\varepsilon_t^2 = |\tilde{C}_t|\eta_t^2$ reflects the impact of innovations for option

prices at time t. The GARCH(1,1) process here follows from a Wold decomposition motivated by our canonical polynomial representation theory of call option. By contrast, (Duan, 1995, pg. 13) used heuristics to construct an elaborate option pricing model with GARCH volatility in his "attempt to link this powerful econometric model with the contingent pricing literature".

Definition 4.1 (Risk factor exposure)**.** Let $\varepsilon_t(\omega)$ be innovations in stochastic option prices, and $\sigma^2_{\tilde{C}_t}$ be a measure of stochastic risk. So that in Equation 4.22 stochastic risk at time t is a function of those two risk factors. Then

A. α_1 is exposure to innovations in option prices; and

B. β_1 is exposure to underlying risk.

In what follows we need the following theorem.

Theorem 4.3 (Convergence of Types)**.** *Let $\triangle$ connote MLE for a given parameter and derived residual. So that $\overset{\triangle}{\alpha}_1$, $\overset{\triangle}{\beta}_1$ are MLE for α_1 and β_1 in the GARCH(1,1) process*

$$\sigma^2_{\tilde{C}_{t+1}} = \alpha_1 \varepsilon_t^2 + \beta_1 \sigma^2_{\tilde{C}_t}$$

Furthermore, let

$$P - \lim_{t\to\infty} \overset{\triangle}{\sigma}{}^2_{\tilde{C}_{t+1}} = \frac{\sigma^2_{\tilde{C}_t}}{1-\alpha_1-\beta_1}$$

Then for any continuous function $g \in C^2(\mathbb{R})$ we have

$$P - \lim\ g(\overset{\triangle}{\alpha}_1, \overset{\triangle}{\beta}_1) = g(\alpha_1, \beta_1)$$

Proof. See (Bollerslev, 1986, Thm. 1 and 2 pp. 310-311) and "convergence of types theorem" in (Durrett, 2005, pg. 156). □

It is clear from Equation 4.22 that we can write innovations in stochastic option prices as a function of the risk factor exposures defined in 4.1. In particular

$$\overset{\triangle}{\varepsilon}_t = \frac{\overset{\triangle}{\sigma}{}^2_{t+1} - \overset{\triangle}{\beta}_1 \overset{\triangle}{\sigma}{}^2_t}{\overset{\triangle}{\alpha}_1} \tag{4.24}$$

On average, MLE estimates of α_1 and β_1 are consistent and efficient. However, an empirical regularity of GARCH(1,1) models is that $\overset{\triangle}{\beta}_1 \gg \overset{\triangle}{\alpha}_1$. That is, call option pricing with stochastic volatility risk exposure implies persistent price risk, while exposure to innovations in option prices suggests that they are comparatively transient. See e.g., (Davidson and MacKinnon, 2004, pg. 579); (Shephard, 1996, pg. 13). Thus, we have the following

Proposition 4.4. *Let ε_t be the innovation in call option prices at time t and developed in period $t-1$, and σ_t be the corresponding price risk. Suppose that the dynamics for call option risk follows a GARCH(1,1) process so that*

$$\overset{\triangle}{\varepsilon}_t = \frac{\overset{\triangle}{\sigma}{}^2_{t+1} - \overset{\triangle}{\beta}_1 \overset{\triangle}{\sigma}{}^2_t}{\overset{\triangle}{\alpha}_1}$$

Then the risk exposure α_1 portends persistent call option risk, and β_1–the exposure to innovations, portends transient shocks to call option risk.

Proof. See Theorem 4.3. □

Remark 4.3. The GARCH(1,1) specification is particularly useful for short term volatility and or risk forecast in a seemingly efficient market[22].

4.1 Empirical pricing kernel estimator for option pricing

The foregoing analysis shows that ARCH and GARCH are *admissible* models for option price fluctuation around trend. However, these fluctuations must decay to reflect long run convergence to the strike price. Specifically, we claim that $\tilde{C}_t$ is well defined by proving that

$$\tilde{C}_{t+1} = \sqrt{|\tilde{C}_t|}\varepsilon_t$$

[22] See (Koverlachuk and Vitayev, 2002, pg. xi) who states:

> The efficient market theory states that it is practically impossible to predict financial markets long-term. However, there is good evidence that short-term trends do exist and programs can be written to find them. The data miners‘ challenge is to find the trends quickly while they are valid, as well as to recognize the time when the trends are no longer effective.

is an admissible decay model for call option price fluctuations. See e.g., (Engle, 2004, pg. 407). Let

$$\tilde{C}_1 = \sqrt{|\tilde{C}_0|}\varepsilon_0 \tag{4.25}$$

Then, by recursion, we get

$$\tilde{C}_k = \varepsilon_{k-1}|\varepsilon_{k-2}|^{-2^{-1}} \dots |\varepsilon_0|^{-2^{-k+1}}|\tilde{C}_0|^{-2^{-k}} \tag{4.26}$$

In which case,

$$\lim_{k\to\infty} \tilde{C}_k = \lim_{k\to\infty} \varepsilon_{k-1}|\varepsilon_{k-2}|^{-2^{-1}} \dots |\varepsilon_0|^{-2^{-k+1}}|\tilde{C}_0|^{-2^{-k}} = 0 \tag{4.27}$$

assuming that the ε-fluctuations are such that they dampen to zero. This is a pseudo Kalman filter result because past error is used for forecasting. See (Box et al., 1994, pg. 165). Because $|\varepsilon| < 1$, the quantities $|\varepsilon_0|^{-2^{-k+1}}|$ are stochastic discount factors. So the derived fluctuations $\tilde{C}_k$ decay and

$$\mathrm{P} - \lim_{k\to\infty} \tilde{C}_k = 0 \tag{4.28}$$

Thus we have just proved the following

Theorem 4.5 (Pricing Kernel Estimator)**.** . *Let $\tilde{C}_t$ be the detrended stochastic option price at time t. Let m_t be the unobservable pricing kernel for call option prices at time t, and η_t be concommittant measurement error. So that $\tilde{m}_t = m_t + \eta_t$ is observed but true m_t and η_t are not. Then the stochastic discount factor or pricing kernel for call option pricing is given by*

$$m_t = \varepsilon_t^{-1} = \frac{1}{\sqrt{|\tilde{C}_t|}\eta_t}$$

Because ε_t and η_t are estimable from ARCH and or GARCH diagnostics we get cross validation for m_t by extrapolating $\overset{\triangle}{m}_t = \tilde{m}_t - \overset{\triangle}{\eta}_t$ by virtue of Theorem 4.3. It is enough to claim that estimation of pricing kernel noise is given by

$$\overset{\triangle}{\eta}_t = \frac{\overset{\triangle}{\varepsilon}_t}{|\tilde{C}_t|} \tag{4.29}$$

So that the signal to noise ratio for the pricing kernel is

$$SNR_{option} = \frac{\overset{\triangle}{\sigma}{}^2_{\tilde{m}_t}}{\overset{\triangle}{\sigma}{}^2_{\eta_t}} \tag{4.30}$$

If $SNR_{option} > 1$, then our model is picking up the "signal" from the true pricing kernel.

4.1.1 Wold decomposition of pricing kernel

According to Wold Dcomposition Theorem 3.1 if $SNR < 1$, then m_t has a long MA representation for trend. If $SNR > 1$, then the deterministic component dominates and the MA representation for trend in short. See (Mills and Markellos, 2008, pg. 118).

Consider the following argument. Let

$$m_t = 1 + u_t \tag{4.31}$$

$$u_t = \theta u_{t-1} + v_t, \quad |\theta| < 1 \tag{4.32}$$

Suppose that η_t is white noise, so that

$$\eta_t = \eta_{t-1} + e_t \tag{4.33}$$

Then

$$\Delta\tilde{m}_t = \Delta m_t + \Delta\eta_t \tag{4.34}$$

$$= (1-\theta L)^{-1}(1-L)v_t + e_t \tag{4.35}$$

where Δ is a difference operator, and L is a lag operator. Under Wold decomposition $\Delta\tilde{m}_t$ is difference stationary. Thus we have the signal

$$z_t = (1-\theta L)^{-1}(1-L)v_t \tag{4.36}$$

and noise e_t. Undeniably, z_t has a moving average (MA) representation. Thus, the "new" SNR is

$$SNR = \frac{\sigma_z^2}{\sigma_e^2} = (1 + \frac{2\theta^2}{1-\theta})\frac{\sigma_v^2}{\sigma_e^2} \tag{4.37}$$

The behavior of θ determines the magnitude of SNR. As long as θ is in the unit circle SNR will be inflated, i.e greater than 1. In particular, if $0 < \theta < 1$ then the signal should be strong. In any case, the decay hypothesis is suppported by Wold decomposition.

5 Conclusion

We propose a solution to an open problem posed in (Kassouf, 1969, pg. 694) by introducing number theory concepts to show that option price formulae depend on algebraic elements in extension fields. In particular, we show that option prices are power laws or polynomials convex in time and volatility of the underlying. To be sure, polynomial expansion of option pricing formulae is not new. What is new is our formal extension to algebraic number theory, and formulation of a canonical representation which produced a class of regenerative option prices, as well as a duality theory for call option with particular applicability to option Greeks estimation. So that, Black and Scholes (1973); Merton (1973) and Kassouf (1969) are special cases of a family of functionally equivalent option pricing formulae that satisfy this criteria. In particular, Kassouf (1969) power law specification is in the class of regenerative polynomial representation for option. Our reduced form polynomial representation of option prices, suggest that in practice synthesis with classic approaches can be used to decompose option prices and provide consistent estimates for risk factor exposure. Of independent interest, is our extension of the analysis to include an empirical specification for the pricing kernel of a call option from residuals in a two-factor risk exposure model. Further research in this are is needed to produce reduced form models as an alternative to the increasingly complex array of exotic option pricing formulae.

6 Appendix

A Proofs

For the benefit of the reader, we reproduce (Clark, 1971, pg. 89)
Proof of Proposition 2.1
Before we begin we need the following

Lemma A.1. . *See (Pollard and Diamond, 1975, Thm. 4.5) . The totality of numbers algebraic over a field F forms a field.*

Proof. Let α, β be algebraic over F, $\alpha \neq 0$ $\beta \neq 0$. We need to show that

$$\alpha+\beta,\ \alpha-\beta,\ \alpha\beta,\ \frac{\alpha}{\beta} \tag{A.1}$$

are algebraic over F. Let $f(x)$ and $g(x)$ be minimal polynomials for α and *beta* over F, respectively. Furthermore, define

$$h_1(x) = \prod_{i=1}^{k}\prod_{j=1}^{n}(x-\alpha_i-\beta_j) \tag{A.2}$$

$$h_2(x) = \prod_{i=1}^{k}\prod_{j=1}^{n}(x-\alpha_i\beta_j) \tag{A.3}$$

Undeniably, h_1 and h_2 are polynomials over F. Hence $\alpha+\beta$ and $\alpha\beta$ are algebraic because there exists roots $\alpha_1+\beta_1$ and $\alpha_1\beta_1$ for the respective equations. Additionally, the relations hold for $h_1(-x)$, $h_2(-x)$. Since $-\beta$ satisfies $h(-\beta)=0$ it is algebraic over F. So the sum $\alpha+(-\beta)=\alpha-\beta$ is algebraic. Let m be the degree of h_2. Then $\frac{1}{\beta}$ satisfies $x^m h_2(\frac{1}{x})$, so it is algebraic. Similarly, $\alpha\frac{1}{\beta}$, is a product of algebraic roots and is thus algebraic. Therefore, the prerequisite conditions for a field are satisfied. □

Proof. (Clark, 1971, pg. 89) Since $F(\alpha)$ is a field that contains α it miust contain the elements $1,\alpha,\alpha^2,\ldots,\alpha^{n-1}$. Consequently, it is a vector space which contains polynomials of the form

$$f(\alpha) = c_0 + c_1\alpha + c_2\alpha^2 + \ldots + c_{n-1}\alpha^{n-1}$$

with coefficients $c_i \in F,\ \ i = 0, 1, \ldots, n-1$. Let

$$X = \{f \mid f(\alpha) = c_0 + c_1\alpha + c_2\alpha^2 + \ldots + c_{n-1}\alpha^{n-1}, \quad c_i \in F,\ \ i = 0, 1, \ldots, n-1\}$$

Thus, X is a vector space over F spanned by

$$\mathscr{V} = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\} \tag{A.4}$$

We claim that $\mathscr{V}$ is linearly independent over F. If there was a nontrivial linear relation over F, depicted by

$$c_0 + c_1\alpha + c_2\alpha^2 + \ldots + c_{n-1}\alpha^{n-1} = 0 \tag{A.5}$$

then α would be a root of the polynomial g over F given by

$$\tag{A.6}$$

$$g(\alpha) = c_0 + c_1\alpha + c_2\alpha^2 + \ldots + c_{n-1}\alpha^{n-1} \tag{A.7}$$

However, the degree g is less than n, and by hypothesis n is the degree of a minimal polynomial for α over F. This contradition implies that $\mathscr{V}$ is linearly independent and hence it is a basis for X over F.

The rest of the proof requires us to show that X is a field. We use a result by Pollard and Diamond (1975) to replace this part of the proof in Clark (1971). Application of Lemma A.1 completes the proof as required. □

References

Ambramowitz, M. and I. A. Stegun (Eds.) (1972). *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables* (9th Printing ed.). Mineola, N.Y.: Dover Publications, Inc.

Apostol, T. M. (1967). *Calculus* (2nd ed.), Volume I. New York: John Wiley& Sons, Inc.

Benth, F. E., M. Groth, and O. Wallin (2010). Derivative-free Greeks for the Barndorff-Nielsen and Shephard Stochastic Volatility Model. *Stochastics An International Journal of Probability and Stochastic Processes: formerly Stochastics and Stochastics Reports 82*(3), 1744–2516.

Black, F. and M. Scholes (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy 81*(3), 637–654.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics 31*, 307–327.

Box, G. E. P., G. M. Jenkins, , and G. Riensel (1994). *Time Series Forecasting and Control* (3rd ed.). New Jersey: Prentice-Hall, Inc.

Brockwell, P. J. and R. A. Davis (1987). *Time Series: Theory and Methods*. New York, NY: Springer-Verlag.

Brown, D. J. and S. Ross (1988, June). Spanning, Valuation and Options. Cowles Foundation Discussion Paper 873, Yale University, published in *Economic Theory*, 1991, 1(1):3-12.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay (1997). *The Econometrics of Financial Markets*. Princeton, N. J.: Princeton University Press.

Chernov, M. (2003). Empirical Reverse Engineering of The Pricing Kernel. *Journal of Econometrics 116*, 329–364.

Chow, Y. and H. Robbins. A Martingale System Theorem and Applications. *Proc. Fourth Berkeley Symp. on Math. Statist. and Prob.*.

Clark, A. (1971). *Elements of Abstract Algebra* (Dover reprint 1984 ed.). Belmont CA: Wadsworth Publishing Co.

Davidson, R. and J. G. MacKinnon (2004). *Econometric Theory and Methods*. New York: Oxford Univ. Press.

Duan, J. (1995). The GARCH Option Pricing Model. *Mathematical Finance 5*(1), 13–32.

Durrett, R. (2005). *Probability: Theory and Examples* (3rd ed.). Belmont, CA: Thompson Brooks-Cole.

Easley, D., M. O'Hara, and P. S. Srinivas (1998, April). Options Volume and Stock Prices: Evidence on Where Informed Traders Trade. *Journal of Finance 53*(2), 431–465.

Engle, R. (1982, July). Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica 50*(4), 987–1008.

Engle, R. F. (2001). GARCH101: The Use of ARCH/GARCH Models in Applied Econometrics. *Journal of Economic Perspectives 15*(4), 157–168.

Engle, R. F. (2004). Risk and Volatility: Econometric Models and Financial Practice. *Amer. Econ. Rev. 94*(3), 405–420.

Fama, E. and H. Babiak (1968, Dec.). Dividend Policy: An Empirical Analysis. *Journal of American Statistical Society 63*(324), 1132–1161.

Flemming, J., B. Ostidek, and R. E. Whaley (1996). Trading Costs and The Relative Rates of Price Discovery in Stocks, Futures, and Options Markets. *Journal of Futures Markets 16*(4), 353–387.

French, D. W. (1983, Fall). Black-Scholes vs. Kassouf Option Pricing: An Empirical Comparison. *Journal of Business Finance and Accounting 10*, 395–408.

Gikhman, I. I. and A. V. Skorokhod (1969). *Introduction to The Theory of Random Processes* (Dover reprint 1996 ed.). Philadelphia, PA: W. B. Saunders.

Gordon, M. J. (1959, May). Dividends, Earnings, and Stock Price. *Review of Economics and Statistics 41*(2), 99–105.

Grillet, P. A. (2007). *Abstract Algebra* (2nd ed.). Graduate Text in Mathematics. New York: Springer-Verlag.

Grossman, S. J. and J. E. Stiglitz (1980, June). On The Impossibility Of Informationally Efficient Markets. *American Economic Review 70*(3), 393–408.

Haug, E. G. and N. N. Taleb (2008). Why We Have Never Used the Black-Scholes-Merton Option Pricing Formula. *Wilmott Magazine 1*, 72–79.

Heston, S. (1993). A Closed-Form Solution for Options with Stochastic Volatility Application to Bond and Currency Options. *Review of Financial Studies 6*(2), 327–343.

Hilbert, D. (1998). *The Theory of Algebraic Number Fields*, Volume English translation of 1898 German edition. New York: Springer-Verlag.

Huang, C. and R. H. Litzenberger (1988). *Foundations of Financial Economics*. Englewoods Cliffs, New Jersey: Prentice-Hall, Inc.

Hull, J. (2006). *Options, Futures, and Other Derivatives* (6th ed.). New Jersey: Prentice-Hall, Inc.

Hull, J. and A. White (1987, June). The Pricing of Options on Assets with Stochastic Volatility. *Journal of Finance 42*(2), 281–300.

Jacobson, N. (1951). *Lectures in Abstract Algebra*, Volume I of *Graduate Text in Mathematics*. New York: Springer-Verlag.

Karatzas, I. and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. New York, N.Y.: Springer-Verlag.

Kassouf, S. T. (1968, May). Stock Price Random Walks: Some Supporting Evidence. *Review of Economics and Statistics 50*(2), 275–278.

Kassouf, S. T. (1969, Oct.). An Econometric Model for Option Price with Implications for Investor Audacity. *Econometrica 37*(4), 685–694.

Kassouf, S. T. (1976). The Lag Structure of Option Price. *Journal of Econometrics 4*, 303–310.

Kolmogorov, A. N. (1956). *Grundbegriffe der Wahrscheinlichkeitrechnung ("Foundations of The Theory of Probability")* (2nd English ed.), Volume English translation of 1933 German edition. New York: Chelsea Publishing Co.

Koverlachuk, B. and E. Vitayev (2002). *Data Mining in Finance: Advances in Relational and Hybrid Methods*. The Kluwer International Series in Engineering and Computer Science. Moscow: Kluwer Academic Publishers.

Madan, D. B. and F. Milne (1991). Options Pricing With V. G. Martingale Components. *Mathematical Finance 1*(4), 39–55.

Madan, D. B. and E. Seneta (1990). The Variance Gamma Model for Share Market Returns. *Journal of Business 63*(4), 511–524.

McCullagh, P. and J. A. Nelder (1989). *Generalized Linear Model* (2nd ed.). Number 37 in Monographs in Statistics and Applied Probability. New York: Chapman & Hall.

Merton, R. C. (1973). A Theory of Rational Options Pricing. *Bell Journal of Economics and Management Science 4*(1), 141–183.

Mills, T. C. and T. N. Markellos (2008). *The Econometric Modeling of Financial Time Series* (3rd ed.). New York: Cambridge University Press.

Pan, J. and A. M. Poteshman (2006, Feb.). The Information in Option Volume for Future Stock Prices. *Review of Financial Studies 19*(3), 871–908.

Passarelli, D. (2008). *Trading Option Greeks*. New York: Bloomberg Press.

Pollard, H. and H. G. Diamond (1975). *The Theory of Algebraic Numbers* (2nd ed.). Number 9 in Carus Mathematical Monograph. Washington, D.C.: Mathematical Association of America.

Rao, C. R. (1973). *Linear Statistical Inference and Its Applications* (2nd ed.). Wiley Series in Probability and Statistics. New York, N. Y.: John Wiley & Sons, Inc.

Reed, M. and B. Simon (1980). *Modern Methods of Mathematical Physics* (Revised Enlarged ed.), Volume Functional Analysis. New York: Academic Press, Inc.

Scott, L. O. (1987, Dec). Option Pricing When the Variance Change Randomly:Theory, Estimation and Application. *Journal of Financial and Quantitative Analysis 22*(4), 419–438.

Shephard, N. (1996). *Time Series Models in Econometrics, Finance and Other Fields*, Chapter Statistical Aspects of ARCH and Stochastic Volatility, pp. 1–67. London: Chapman & Hall.

Snell, J. L. (1952, Sept.). Applications To Martingale System Theorems. *Trans. Amer. Math. Soc. 73*(2), 293–312.

Wang, Y. and R. Caflish (2010, May). Fast Computation of Upper Bounds for American-style Options without Nested Simulation. Dep't of Mathematics, Univ. California-Los Angeles, available at SSRN http://ssrn.com/abstract=1612336.

Weisberg, S. (2005). *Applied Linear Regression* (3rd ed.). Wiley Series in Probability and Statistics. New York: John Wiley & Sons, Inc.

Whaley, R. E. (2000). The Investor Fear Gauge: Explication of the CBOE VIX. *Journal of Portfolio Management 26*(3), 12–17.